Celestial Mechanics: Application of Kepler's Laws and Spherical Trigonometry

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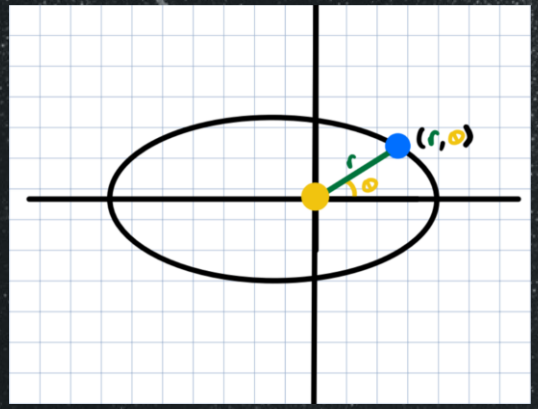
Kepler’s Laws

Kepler’s First Law

Planets orbit in elliptical paths
with their star as a focus.

Kepler's First Law

- Planets orbit around their host star
- These orbits are elliptical in nature
- The star occupies one focus of this ellipse



The equation for the planet's orbit

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

r is the distance from the planet to the star, *a* is the length of the orbit's semi major axis and *e* is the eccentricity of the orbit.

Kepler’s Second Law

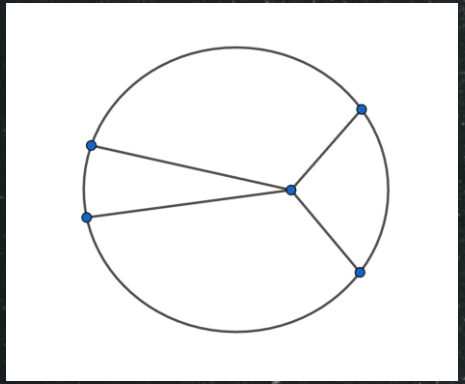
The radius from the star to the planet sweeps out equal areas in equal amounts of time.

Kepler's Second Law

As a planet moves around its star, the radius between the bodies sweeps out an area which remains constant per unit time.

The change in area is given by:

$$\mathrm{d}A = \frac{1}{2} r^2 \left(\frac{h}{r^2} \right) \mathrm{d}t = \frac{h}{2} \mathrm{d}t$$



Where h is a constant such that $h^2 = GMa(1 - e^2)$ and r is the distance between the planet and its star.

Kepler’s Third Law

The square of the period of a planet's orbit is proportional to the cube of its semi-major axis.

Kepler's Third Law

- The area of an ellipse is πab
- a is the length of the semi-major axis and b is the length of the semi-minor axis.
- Let T represent the period of the planet's orbit

The area can be calculated by integrating dA from 0 to T. From Kepler's 2nd Law, dA = (h/2)dt. After integrating and solving for T^2, the result is:

$$T^2 = \frac{4\pi^2 a^2 b^2}{h^2}$$

Since $h^2 = GMa(1 - e^2)$, the final result is:

$$T^2 = \frac{4\pi^2}{GM} a^3$$

Conservation of Energy

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$$

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

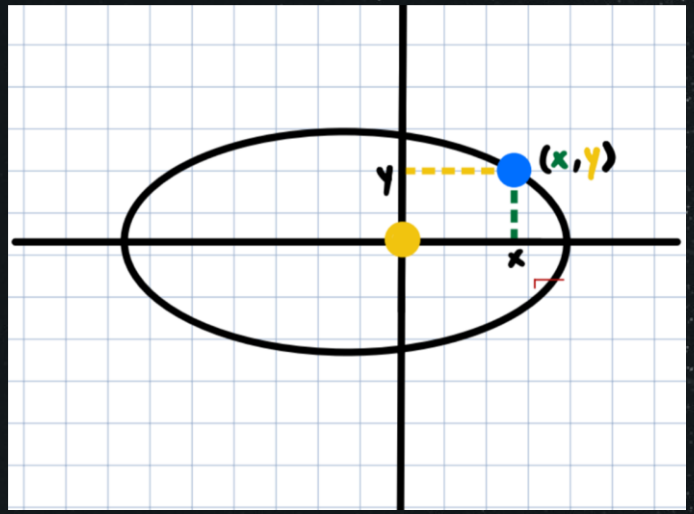
$$E = -\frac{GMm}{2a}$$

Energy is conserved.

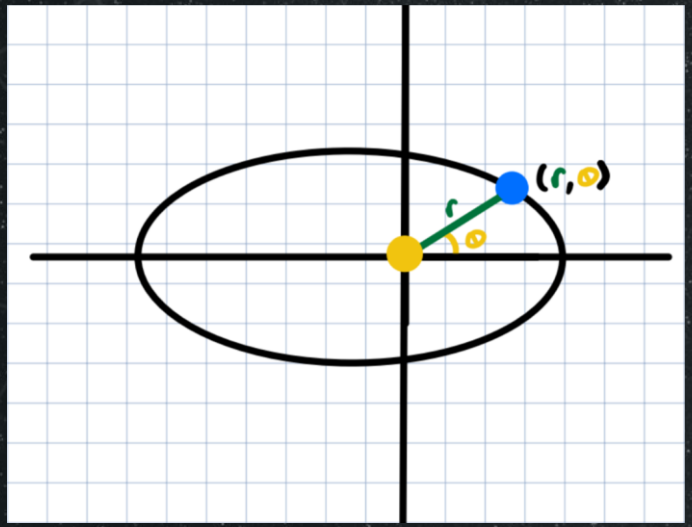
03

Ellipse Geometry

Ellipse Geometry



$$\frac{(x+c)^2}{a^2}+\frac{y^2}{b^2}=1$$



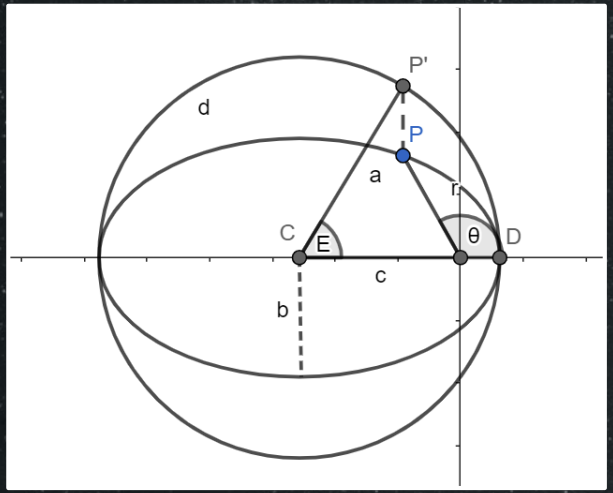
$$r=\frac{a(1-e^2)}{1+e\cos\theta}$$

Connecting E to θ

$$\cos E = \frac{\cos\theta + e}{1 + e\cos\theta}$$

$$r\sin\theta = b\sin E$$

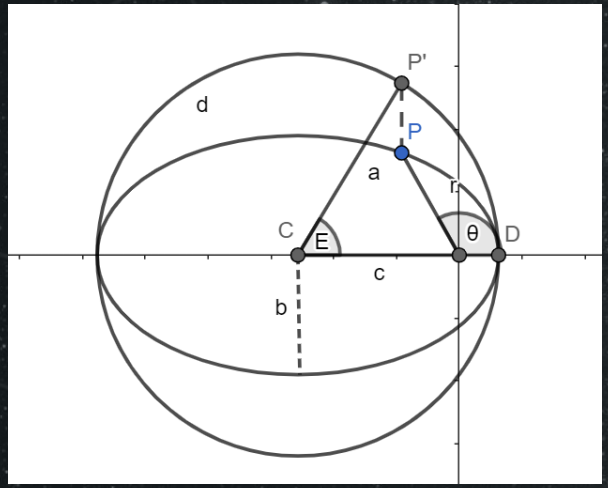
$$\tan\left(\frac{E}{2}\right) = \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\left(\frac{\theta}{2}\right)$$



Showing that $\frac{dM}{dt}$ is constant

$$M = E - e\sin E$$

$$\frac{dM}{dt} = \frac{h}{ab} = \frac{2\pi}{T}$$



Now we've connected time to θ

04

Spherical Trigonometry

Spherical Trigonometry

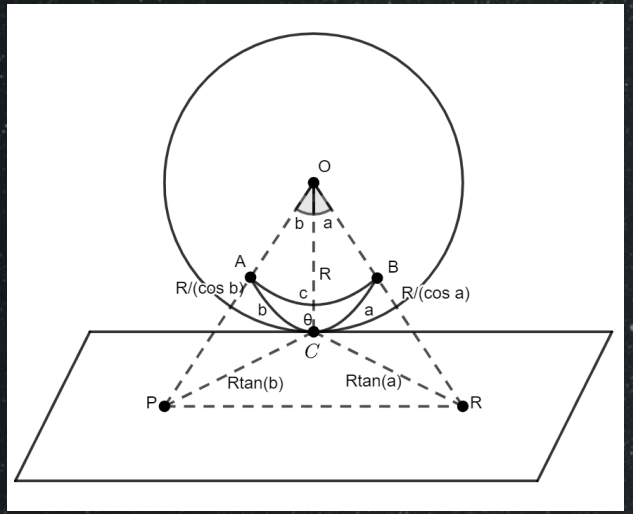
- Geometry on a spherical surface has different rules
- Shortest path between two points lies on a “Great Circle”
- Spherical triangles are formed by these arcs

Spherical Law of Sines

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Spherical Law of Cosines

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$



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Celestial Sphere

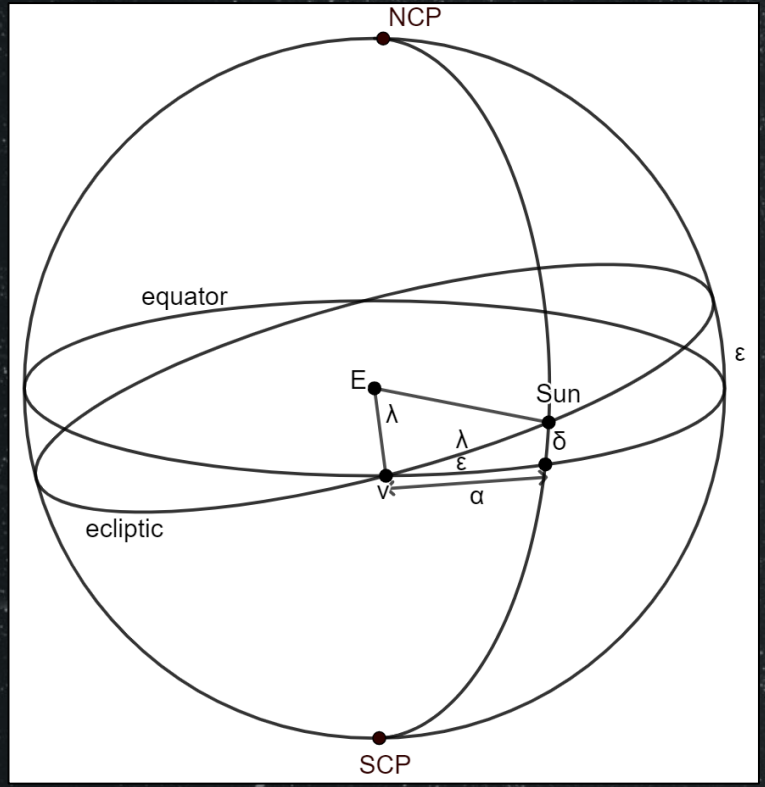
A Shift in Reference Frame.....

Principles

- An apparent sphere formed by the sky (and its other half below the horizon)
- Rotates East to West
- Longitude, Latitude → Right Ascension, Declination

Assumptions

- Arbitrarily large distances



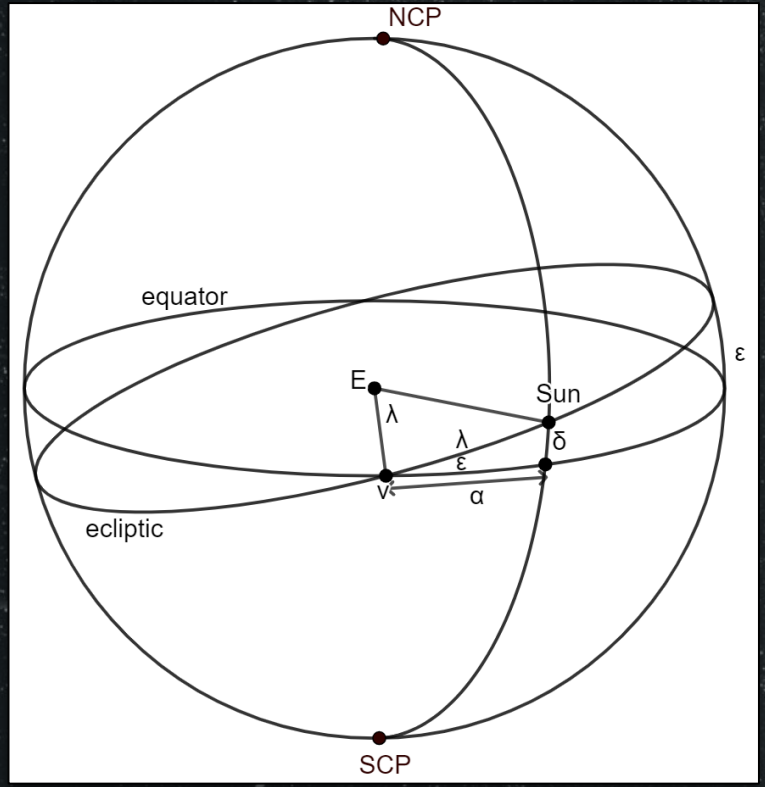
Points of Reference

Equatorial Coordinates

- Celestial Equator
- Vernal Equinox
- Celestial Poles

Ecliptic Coordinates

- Ecliptic
- Vernal Equinox
- Ecliptic Poles



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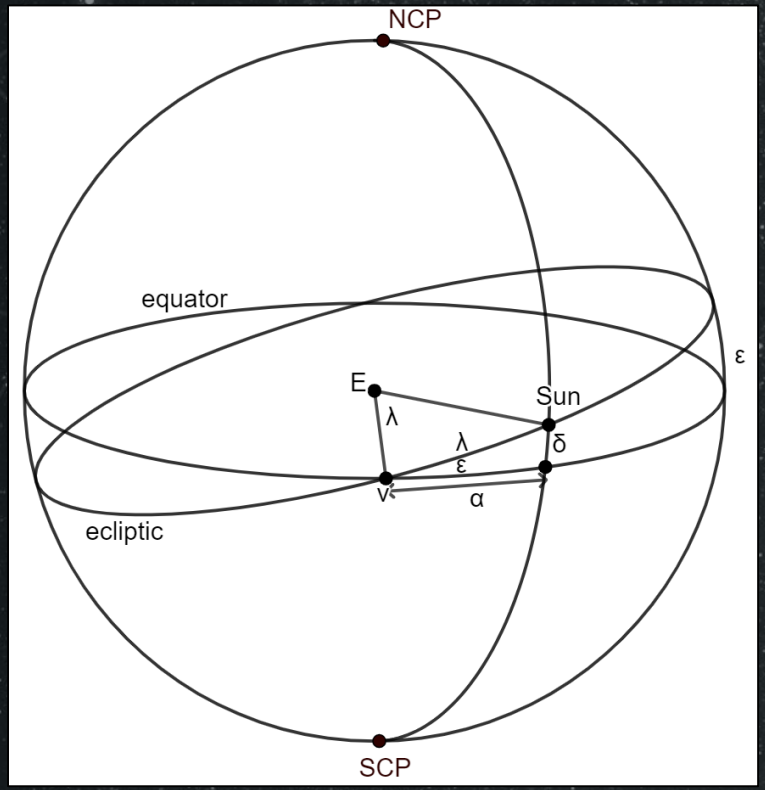
Rise & Set Times

Sunrise and Sunset

- declination determines the path of the sun in the sky
- one hemisphere visible from a given point on the Earth
  - time the sun spends in that hemisphere is the time the sun is in the sky
- highest point is noon
  - subtract off for sunrise
  - add on for sunset



Saturn Rise and Saturn Set

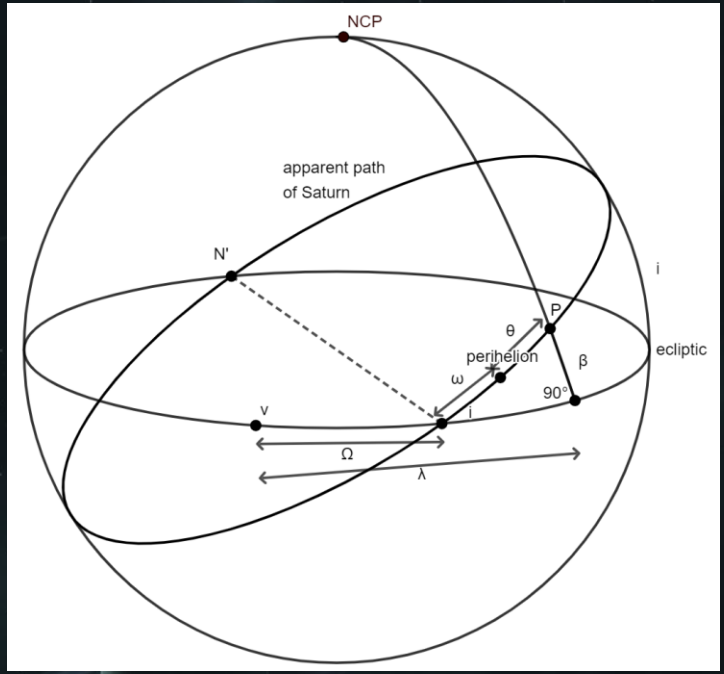
- Given θ for Saturn, we can calculate "ecliptic" coordinates

⇓

- Given ecliptic coordinates, we can calculate Right Ascension and Declination

⇓

- Given RA and Decl., we can calculate time spent in the sky.



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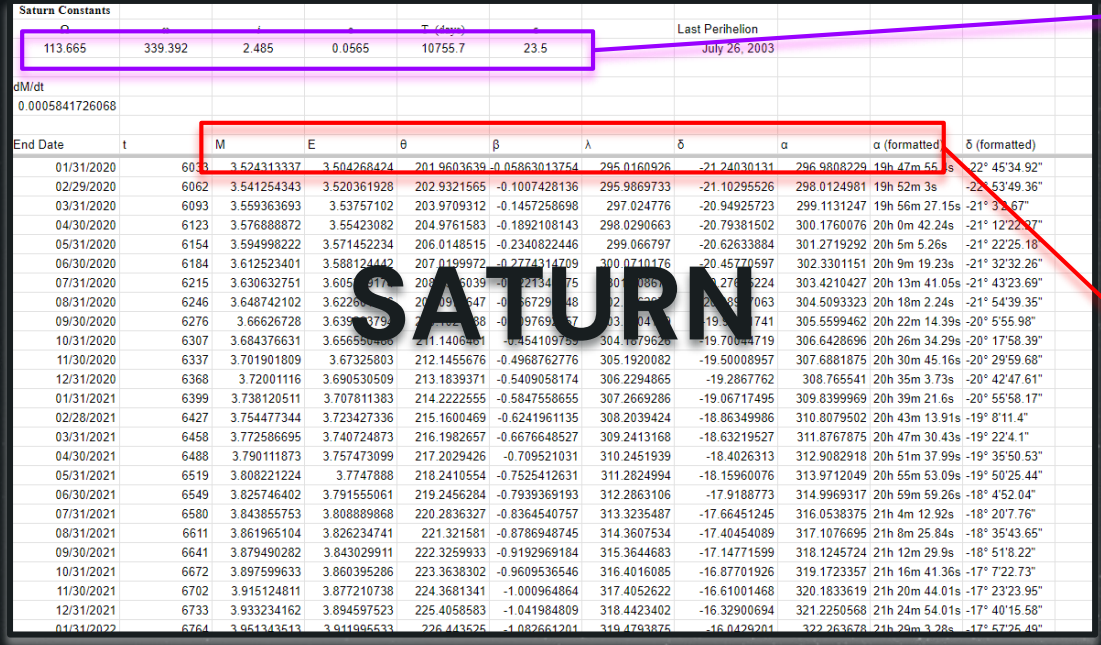
Machine Learning – Error Analysis

Generating Data: Kepler's Laws



Saturn Constants

Ω	i	ω	a	T (days)	e	Last Perihelion
113.665	339.392	2.485	0.0565	10755.7	23.5	July 26, 2003

dM/dt
0.0005841726068

End Date	t	M	E	θ	β	λ	δ	α	α (formatted)	δ (formatted)
01/31/2020	6038	3.524313337	3.504268424	201.9603639	-0.05863013754	295.0160926	-21.24030131	296.9808229	19h 47m 55.39s	-22° 45'34.92"
02/29/2020	6062	3.541254343	3.520361928	202.9321565	-0.1007428136	295.9869733	-21.10295526	298.0124981	19h 52m 3s	-22° 53'49.36"
03/31/2020	6093	3.559363693	3.53757102	203.9709312	-0.1457258698	297.024776	-20.94925723	299.1131247	19h 56m 27.15s	-21° 3'[illegible].67"
04/30/2020	6123	3.576888872	3.55423082	204.9761583	-0.1892108143	298.0290663	-20.79381502	300.1760076	20h 0m 42.24s	-21° 12'22.97"
05/31/2020	6154	3.594998222	3.571452234	206.0148515	-0.2340822446	299.066797	-20.62633884	301.2719292	20h 5m 5.26s	-21° 22'25.18"
06/30/2020	6184	3.612523401	3.588124442	207.0199972	-0.2774314709	300.0710176	-20.45770597	302.3301151	20h 9m 19.23s	-21° 32'32.26"
07/31/2020	6215	3.630632751	3.605[illegible]	208[illegible]039	-0.[illegible]2134[illegible]75	30[illegible]0836	-20.276[illegible]224	303.4210427	20h 13m 41.05s	-21° 43'23.69"
08/31/2020	6246	3.648742102	3.622[illegible]	209[illegible]647	-0.[illegible]56729[illegible]48	3[illegible]2	-20.[illegible]897063	304.5093323	20h 18m 2.24s	-21° 54'39.35"
09/30/2020	6276	3.66626728	3.639[illegible]379[illegible]	[illegible]8	-0.[illegible]97691[illegible]	3[illegible]041	-19.9[illegible]7741	305.5599462	20h 22m 14.39s	-20° 5'55.98"
10/31/2020	6307	3.684376631	3.656550488	211.1406461	-0.454109759	304.1879626	-19.70044719	306.6428696	20h 26m 34.29s	-20° 17'58.39"
11/30/2020	6337	3.701901809	3.67325803	212.1455676	-0.4968762776	305.1920082	-19.50008957	307.6881875	20h 30m 45.16s	-20° 29'59.68"
12/31/2020	6368	3.72001116	3.690530509	213.1839371	-0.5409058174	306.2294865	-19.2867762	308.765541	20h 35m 3.73s	-20° 42'47.61"
01/31/2021	6399	3.738120511	3.707811383	214.2222555	-0.5847558655	307.2669286	-19.06717495	309.8399969	20h 39m 21.6s	-20° 55'58.17"
02/28/2021	6427	3.754477344	3.723427336	215.1600469	-0.6241961135	308.2039424	-18.86349986	310.8079502	20h 43m 13.91s	-19° 8'11.4"
03/31/2021	6458	3.772586695	3.740724873	216.1982657	-0.6676648527	309.2413168	-18.63219527	311.8767875	20h 47m 30.43s	-19° 22'4.1"
04/30/2021	6488	3.790111873	3.757473099	217.2029426	-0.709521031	310.2451939	-18.4026313	312.9082918	20h 51m 37.99s	-19° 35'50.53"
05/31/2021	6519	3.808221224	3.7747888	218.2410554	-0.7525412631	311.2824994	-18.15960076	313.9712049	20h 55m 53.09s	-19° 50'25.44"
06/30/2021	6549	3.825746402	3.791555061	219.2456284	-0.7939369193	312.2863106	-17.9188773	314.9969317	20h 59m 59.26s	-18° 4'52.04"
07/31/2021	6580	3.843855753	3.808889868	220.2836327	-0.8364540757	313.3235487	-17.66451245	316.0538375	21h 4m 12.92s	-18° 20'7.76"
08/31/2021	6611	3.861965104	3.826234741	221.321581	-0.8786948745	314.3607534	-17.40454089	317.1076695	21h 8m 25.84s	-18° 35'43.65"
09/30/2021	6641	3.879490282	3.843029911	222.3259933	-0.9192969184	315.3644683	-17.14771599	318.1245724	21h 12m 29.9s	-18° 51'8.22"
10/31/2021	6672	3.897599633	3.860395286	223.3638302	-0.9609536546	316.4016085	-16.87701926	319.1723357	21h 16m 41.36s	-17° 7'22.73"
11/30/2021	6702	3.915124811	3.877210738	224.3681341	-1.000964864	317.4052622	-16.61001468	320.1833619	21h 20m 44.01s	-17° 23'23.95"
12/31/2021	6733	3.933234162	3.894597523	225.4058583	-1.041984809	318.4423402	-16.32900694	321.2250568	21h 24m 54.01s	-17° 40'15.58"
01/31/2022	6764	3.951343513	3.911995533	226.443525	-1.082661201	319.4793875	-16.0429201	322.263678	21h 29m 3.28s	-17° 57'25.49"

SATURN

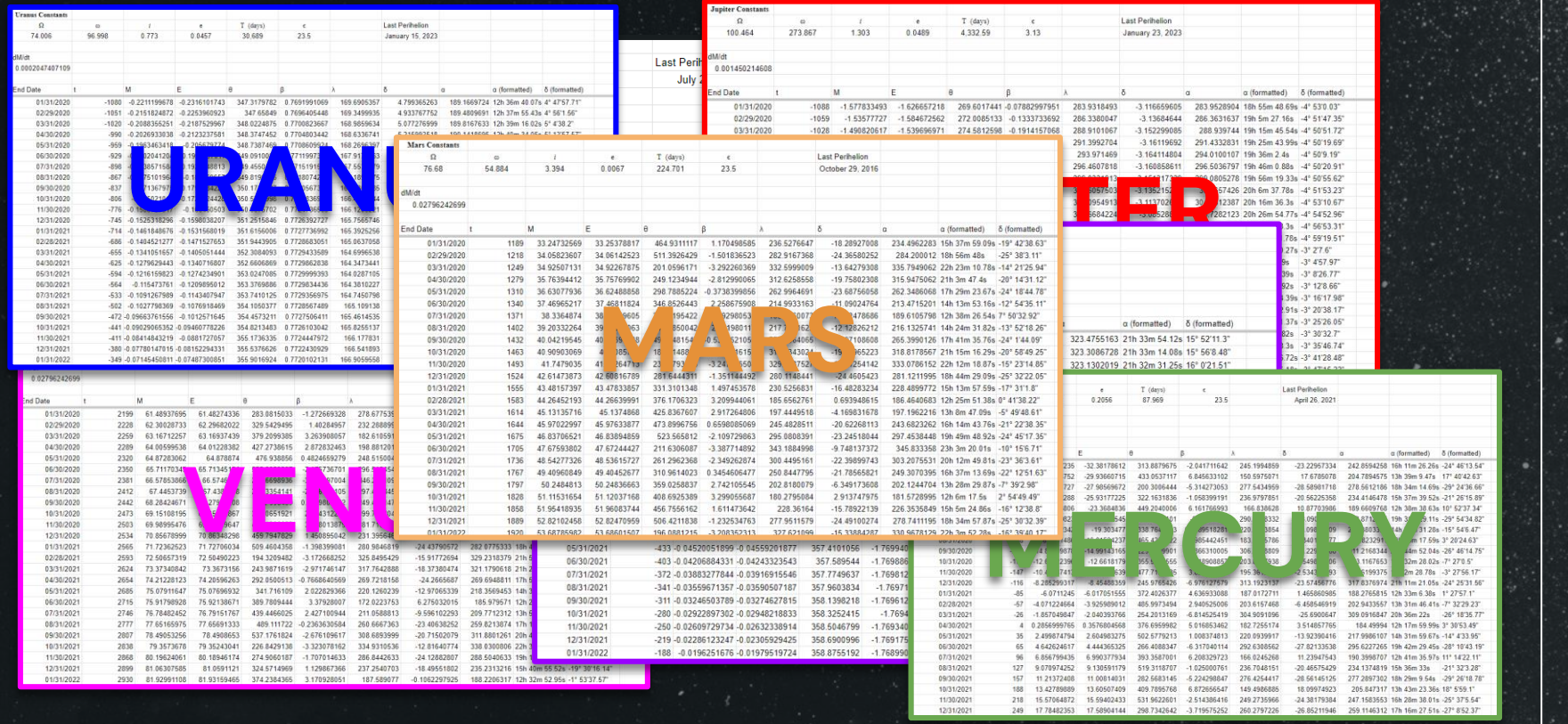
INPUTTED CONSTANTS

For each planet, constants were found and inputted

PROGRAMMED CALCULATIONS

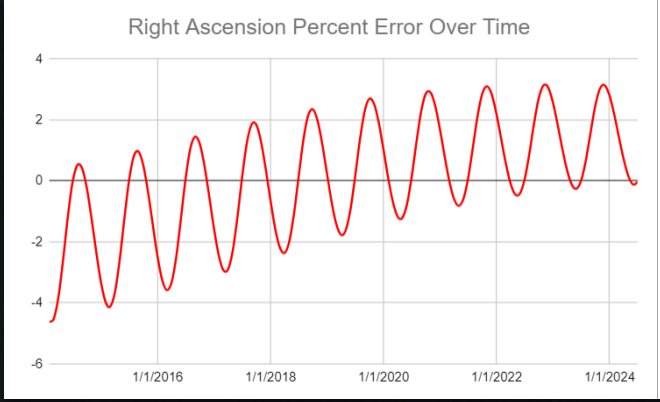
Used mathematical equations derived previously to generate hundreds of data values from many times

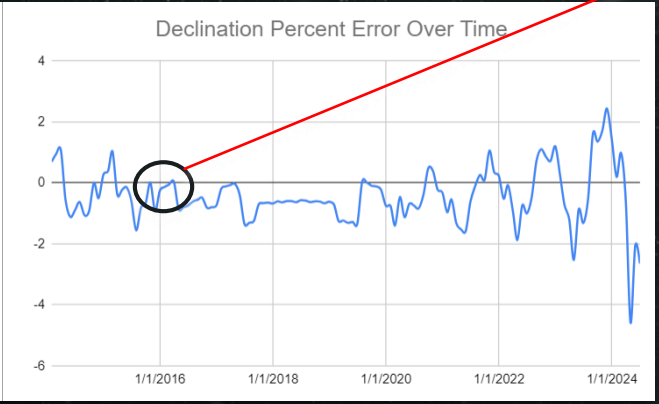
Generating Data: Kepler's Laws



Application: Predicting Saturn's Orbit

Error Calculation





MINOR FLUCTUATION AROUND 0%

Explained by:

Gravitational Forces from Other Objects (ex. Jupiter)

Effects of General Relativity

Linear Regression Models

- *Generated celestial coordinates for Saturn for each day 01/01/2014-05/31/2024*
- *Obtained actual values from an ephemeris*
- *Determined linear relationships between calculated data and actual observations (linearized error)*
- *Tested regression models on test data*
- *Compared models' predictions to ephemeris data*

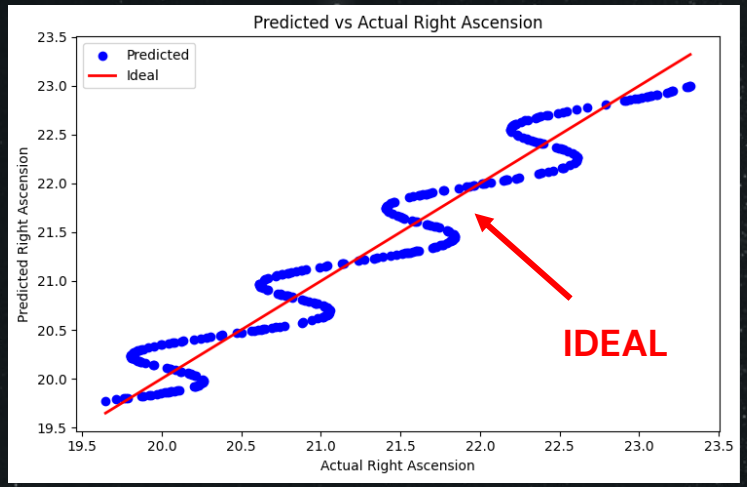
```
model_alpha = LinearRegression()
model_alpha.fit(X_train, y_alpha_train)
```

```
model_delta = LinearRegression()
model_delta.fit(X_train, y_delta_train)
```

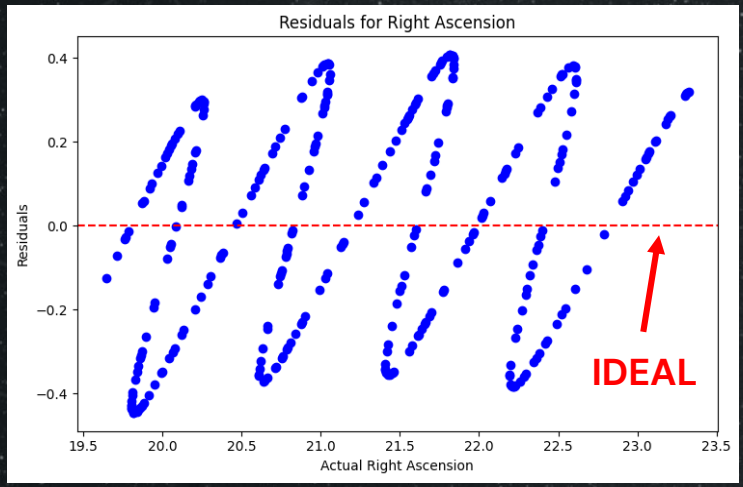
```
y_alpha_pred = model_alpha.predict(X_test)
y_delta_pred = model_delta.predict(X_test)
```

Model Results

Right ascension (α)



Predictions vs Actual Ephemeris Data

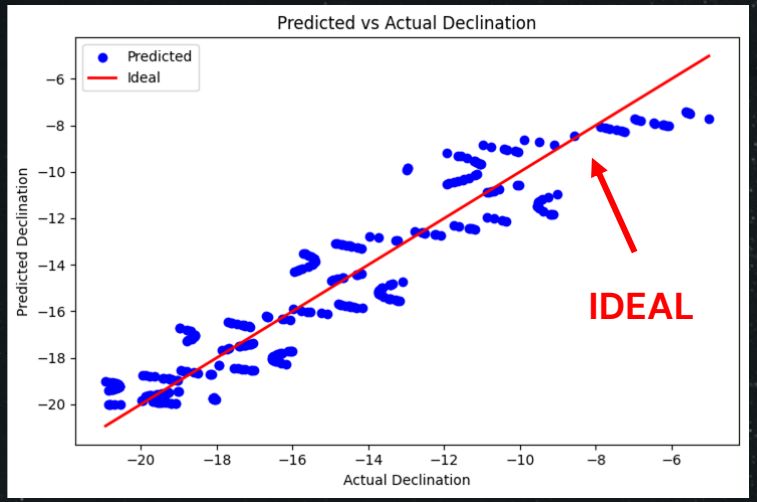


Residual Plot

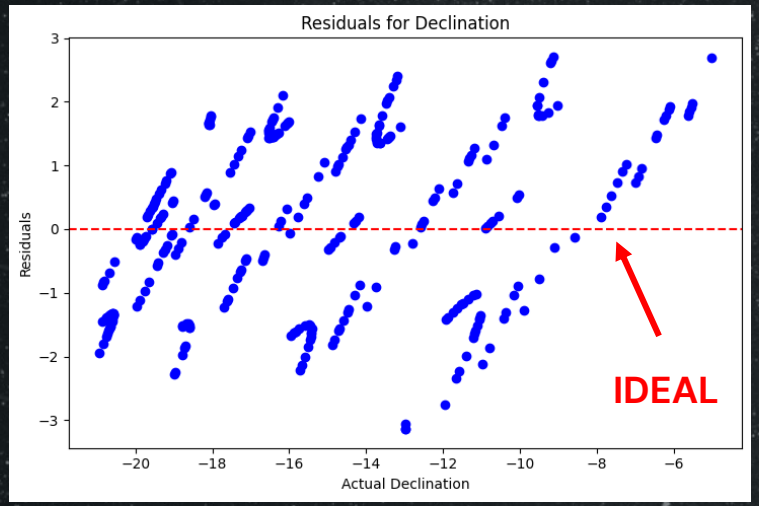
R^2: 0.9276

Model Results

Declination (δ)



Predictions vs Actual Ephemeris Data



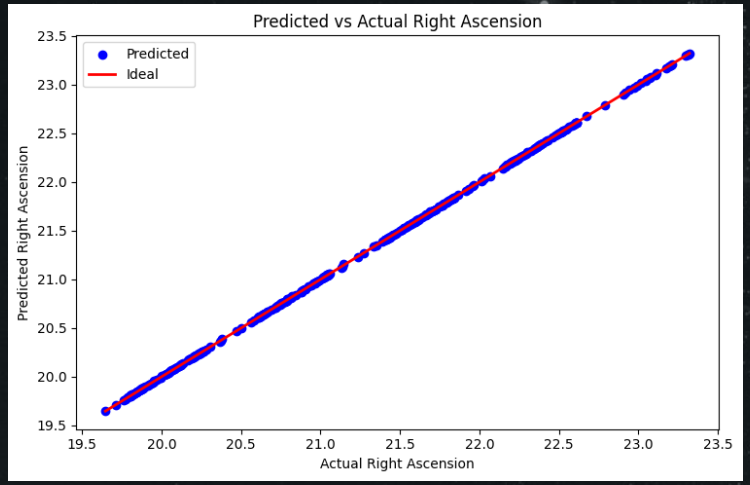
Residual Plot

R^2: 0.8895

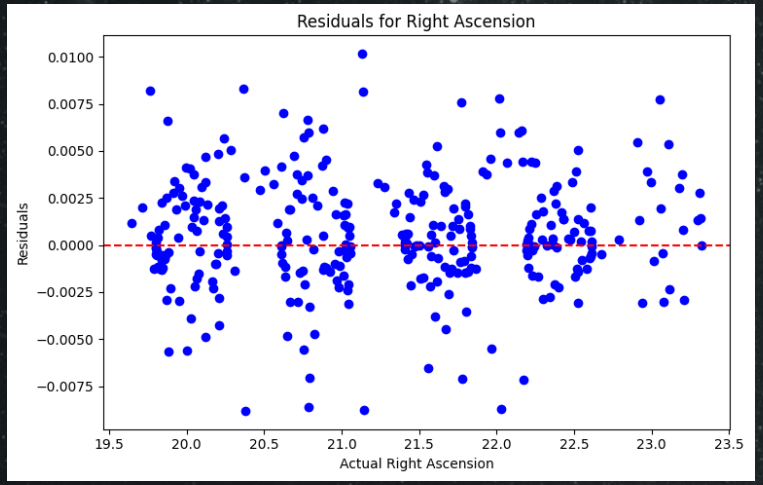
Random Forest Models

Using more complex models to account for underlying periodic trends

Right ascension (α)



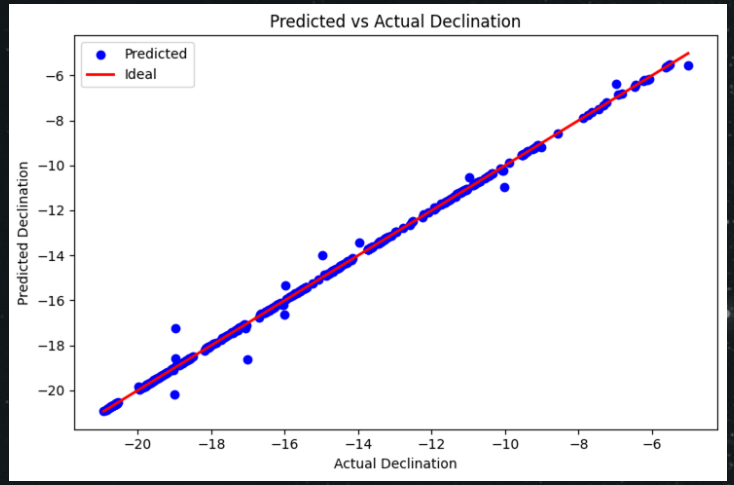
Predictions vs Actual Ephemeris Data



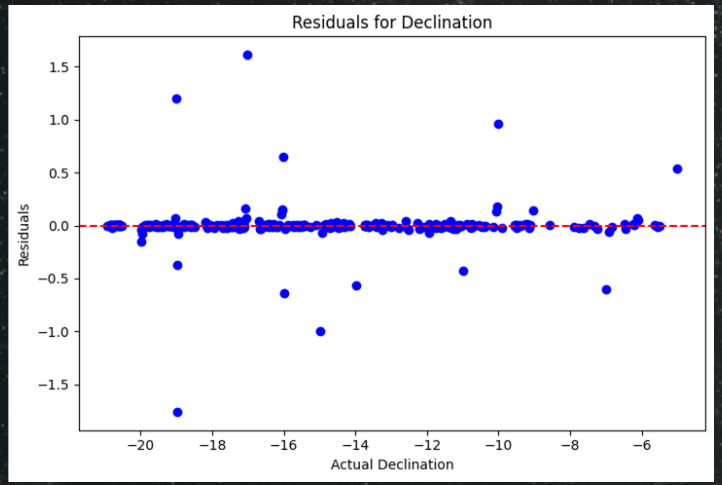
Residual Plot

R^2: 0.99

Declination (δ)



Predictions vs Actual Ephemeris Data



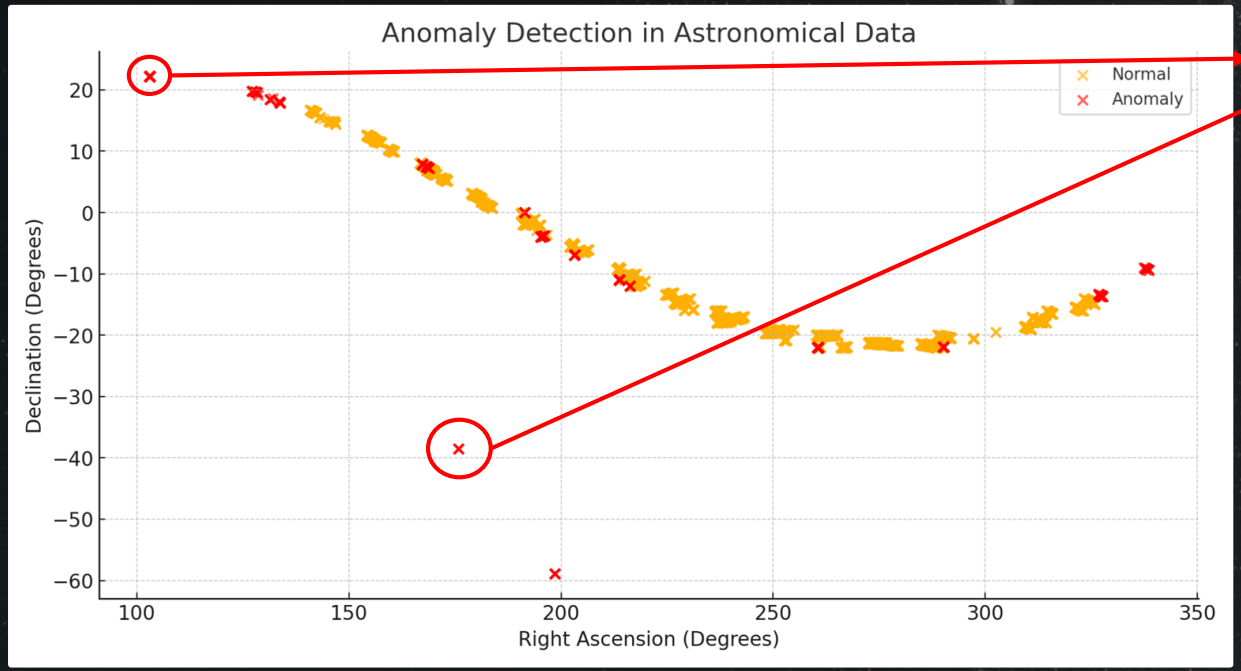
Residual Plot

R^2: 0.99

09

Machine Learning – Anomaly Detection

One-Class Support Vector Machines



ANOMALIES DETECTED

Radial Basis Function Kernel

Explained by:

Unusual celestial events

Observational or recording error

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Conclusion

Machine Learning Integration

Linear regression and random forest models increased prediction precision.

One-Class SVM detected anomalies in planetary orbits, indicating unusual events or errors

Key Insights

Classical mechanics and modern techniques to enhance astronomical predictions.

Future work could involve training more complex models for improved accuracy.

Extend techniques to predict other future celestial phenomena and objects.

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Thank You!

Q & A

In loving spirit of William Colangelo III & Tyler Harms