CELESTIAL MECHANICS: APPLICATION OF KEPLER'S LAWS AND SPHERICAL TRIGONOMETRY

New Jersey Governor's School in the Sciences

William Colangelo III, Adrien Cristian, Stefano D'Agostino, Mayank Deoras, Rishay Gupta, Tyler Harms, Jeffrey Jiang, David Ji, Aditya Kirubakaran, Krish Shah, Timothy Torubarov, Kevin Zhang

July 26, 2024

Advisor: Steve Surace
Assistant: Clifford Wijaya

ABSTRACT

Astronomy has evolved from qualitative observations to a precise science, founded on the basic laws of gravitation. Isaac Newton's law of universal gravitation and Kepler's laws of planetary motion remain vital for predicting celestial movements. This paper applies these principles to calculating the positions of Earth and and other celestial bodies on any specific date, the exact time of sunset and sunrise, and predicting the times at which celestial bodies become visible each day. By deriving equations using Newton's and Kepler's laws and exploring geometric representations of elliptical orbits, the relevance of Newtonian mechanics in contemporary astronomy is highlighted. This study provides practical insights into the motion of planets and a very close approximation of the orbits of celestial objects in the solar system.

1 INTRODUCTION

Looking up at the night sky, one could see hundreds of millions of specks of light, with trillions more hidden from sight. The universe has always been of interest to human cultures, being the focus of thousands of years of study. Astronomy, one of the oldest disciplines of the natural sciences, has evolved greatly by moving from simple qualitative observations to an advanced quantitative science over the last 5 centuries. This evolution was significantly influenced by the work of Isaac Newton and Johannes Kepler, whose laws of gravitation and planetary motion, respectively, have revolutionized the way the night sky is analyzed.

Newton discovered a mathematical relationship between the distance and masses of two objects and the forces between them. Despite the advances in understanding of gravitation with Einstein's theory of relativity, Newton's laws remain crucial for predicting the position of celestial bodies in the sky. Alongside those of Newton, Kepler's laws offer specific insights into analyzing elliptical orbits. [1]

In this paper, Kepler's laws are derived beginning with Newton's Universal Law of Gravitation and the principles of calculus. Later, by deriving a polar representation for elliptical orbits, a relationship between the time and position of a planet in its orbit was determined.

Then, through an investigation of spherical geometry, a way to locate the positions of celestial objects as perceived from Earth was derived. This unique system for determining the position of an object over time was applied to the Earth for a specific date, alongside an investigation into Saturn's position.

The paper outlines a straightforward routine to accurately predict the locations and times of celestial objects and events; however, this study also integrates the contemporary technique of machine learning to refine predictions of celestial positions, highlighting the ongoing relevance of classical mechanics in modern astronomical research.

2 KEPLER'S LAWS

2.1 Preliminaries

In 1687, Sir Isaac Newton published the groundbreaking book "Philosophiae Naturalis Principia Mathematica." In addition to establishing Newton's three laws of motion, "Philosophiae Naturalis Principia Mathematica" also proposed a law of universal gravitation.

Universal Law of Gravitation. (Newton [1]) Any two objects in the universe are attracted to each other with a force that is inversely proportional to the square of the distance between the centers of the two objects and directly proportional to the product of their masses. Mathematically, this means that

$$F_g = \frac{GMm}{r^2}$$

where F_g is the force of gravity, M and m are the masses of the objects, $G = 6.67 \quad 10^{\;11} \frac{N\,m^2}{kg^2}$ is the gravitational constant, and r is the distance between the two objects.

Newton's Law of Universal Gravitation can be used to derive three statements known collectively as Kepler's Laws.

Consider a planet of mass m orbiting around a star of mass M. Let r be the distance from the planet to the star, and let θ be the angle formed between the radius from the star to the planet and the radius from the star to the planet's perihelion, as shown in Figure 1.

Examining the x and y components of the planet's acceleration due to the Universal Law of Gravitation, it can be seen that

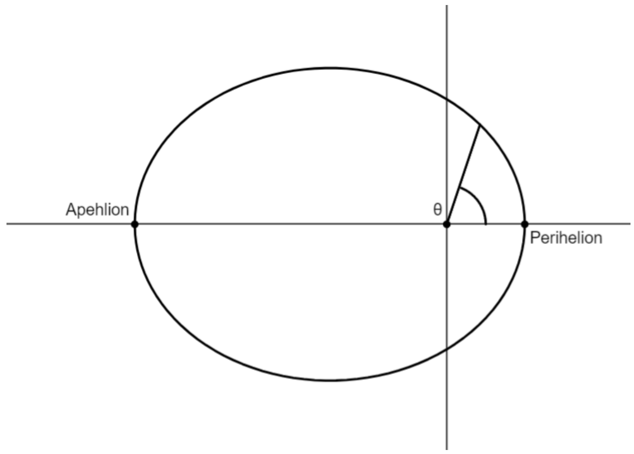

Figure 1: Angle from Perihelion

$$
\begin{aligned}
\frac{GM}{r^2}\cos\theta &= \frac{d^2}{dt^2}(r\cos\theta) \\
&= \cos\theta\frac{d^2r}{dt^2} \quad 2\sin\theta\frac{d\theta}{dt}\frac{dr}{dt} \quad r\cos\theta\left(\frac{d\theta}{dt}\right)^2 \quad r\sin\theta\frac{d^2\theta}{dt^2}
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
\frac{GM}{r^2}\sin\theta &= \frac{d^2}{dt^2}(r\sin\theta) \\
&= \sin\theta\frac{d^2r}{dt^2} + 2\cos\theta\frac{d\theta}{dt}\frac{dr}{dt} \quad r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}
\end{aligned}
\tag{2}
$$

Subtracting the product of Equation (1) by $\sin\theta$ from the product of Equation (2) by $\cos\theta$ yields

$$
\begin{aligned}
0 &= 2(\cos^2\theta + \sin^2\theta)\frac{d\theta}{dt}\frac{dr}{dt} + r(\cos^2\theta + \sin^2\theta)\frac{d^2\theta}{dt^2} \\
0 &= 2\frac{d\theta}{dt}\frac{dr}{dt} + r\frac{d^2\theta}{dt^2}.
\end{aligned}
\tag{3}
$$

Substituting $\omega = \frac{d\theta}{dt}$ gives

$$
\begin{aligned}
0 &= 2\omega\frac{dr}{dt} + r\frac{d\omega}{dt} \\
\int \frac{1}{\omega}d\omega &= \int \quad \frac{2}{r}dr \\
\frac{d\theta}{dt} &= \frac{h}{r^2},
\end{aligned} \tag{4}
$$

where h is a constant.

Similarly, adding the product of Equation (1) by $\cos\theta$ to the product of Equation (2) by $\sin\theta$ yields

$$
\frac{GM}{r^2} = \frac{d^2r}{dt^2} \quad r\left(\frac{d\theta}{dt}\right)^2 \tag{5}
$$

Employing the substitution $u = \frac{1}{r}$ and Equation (4), by the chain rule:

$$
\begin{aligned}
GMu^2 &= \frac{d}{dt}\left(\frac{d}{dt}\frac{1}{u}\right) \quad h^2u^3 \\
&= \frac{d}{dt}\left(\quad \frac{1}{u^2}\frac{du}{dt}\right) \quad h^2u^3 \\
&= \frac{d}{dt}\left(\quad h\frac{du}{d\theta}\right) \quad h^2u^3 \\
&= \quad h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} \quad h^2u^3 \\
\frac{GM}{h^2} &= \frac{d^2u}{dt^2} + u
\end{aligned} \tag{6}
$$

The solutions to Differential Equation (6) are of the form

$$
u = k_1\cos\theta + k_2\sin\theta + \frac{GM}{h^2}, \tag{7}
$$

where k_1 and k_2 are constants. Note now that

$$
\frac{du}{d\theta} = \quad k_1\sin\theta + k_2\cos\theta \tag{8}
$$

and further

$$\frac{d^2u}{dt^2} = \quad k_1 \cos\theta \quad k_2 \sin\theta. \tag{9}$$

By definition, when $\theta = 0$, the planet is at its perihelion, so u is maximal. Thus, $\frac{du}{dt} = 0$ and $\frac{d^2u}{dt^2} < 0$ so $k_2 = 0$ and $k_1 > 0$ so

$$\begin{aligned} r &= \frac{1}{\frac{GM}{h^2} + k_1 \cos\theta} \\ &= \frac{\frac{h^2}{GM}}{1 + \frac{h^2}{GM} k_1 \cos\theta}. \end{aligned} \tag{10}$$

2.2 Kepler's First Law

Kepler's First Law describes the shape of a planet's orbital path about its star.

Equation (10) is of the form Equation (25) in Appendix B with

$$\frac{h^2}{GM} = a(1 \quad w^2) \tag{11}$$

and

$$\frac{h^2}{GM} k_1 = e$$

Thus, the orbital path of the planet is an ellipse with its star at one focus.

Kepler's First Law. Planets orbit in elliptical paths with their star at one focus.

2.3 Kepler's Second Law

Kepler's Second Law establishes the invariance of the area that the radius from a planet to its star sweeps out per unit time.

If A denotes the area swept out by the radius from the star to the planet, then $dA = \frac{1}{2} r^2 d\theta$. Thus, by Equation (4)

$$dA = \frac{1}{2} r^2 \left(\frac{h}{r^2} dt \right) = \frac{h}{2} dt. \tag{12}$$

Since $\frac{dA}{dt}$ is constant, the area swept out by the radius from the star to the planet per unit of time is constant.

Kepler's Second Law. The radius from the star to the planet sweeps out equal areas in equal amounts of time.

2.4 Kepler's Third Law

Finally, Kepler's Third Law relates the period of a planet's orbit to the semi-major axis of that orbit.

The area of an ellipse is given by the expression $A = \pi ab$, where a represents the length of the semi-major axis and b represents the length of the semi-minor axis. Let T represent the period of the planet's orbit. Applying Equation (12), it is seen that

$$\begin{aligned} \pi ab &= \int_0^T dA \\ \pi ab &= \int_0^T \frac{h}{2} dt \\ \pi ab &= \frac{h}{2} T \\ \frac{4\pi^2 a^2 b^2}{h^2} &= T^2 \end{aligned}$$

By Equation (11), this suggests that

$$\begin{aligned} T^2 &= \frac{4\pi^2 a^2 b^2}{h^2} \\ &= \frac{4\pi^2 a^2 b^2}{GMa(1 \quad e^2)} \\ &= \frac{4\pi^2 a^2 b^2}{GMa(1 \quad \left(\frac{c}{a}\right)^2)} \\ &= \frac{4\pi^2}{GM} a^3. \end{aligned} \tag{13}$$

Kepler's Third Law. The square of the period of a planet's orbit is proportional to the cube of its semi-major axis.

With Kepler's Laws now proven through the use of Calculus, the new-formed relationships can be used to verified the fundamental laws of the Universe, such as the conservation of Energy.

3 CONSERVATION OF ENERGY

Kepler's Laws and the differential equations on which they are based directly imply the principle of conservation of energy.

The total energy of a planet is given by

$$E = \frac{1}{2}mv^2 \quad \frac{GMm}{r} \tag{14}$$

where v is the velocity of the planet, m is the mass of the planet, M is the mass of the star, and r is the distance from the planet to the star.

The square of the velocity of a planet is given by $v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ with $x = r\cos\theta$ and $y = r\sin\theta$.

Applying these substitutions, it is calculated that

$$\begin{aligned} v^2 &= \left(\frac{d}{dt}r\cos\theta\right)^2 + \left(\frac{d}{dt}r\sin\theta\right)^2 \\ &= \left(\frac{dr}{dt}\cos\theta \quad r\sin\theta\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)^2 \\ &= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \end{aligned} \tag{15}$$

By Equation (10), it is seen that

$$\begin{aligned} \frac{dr}{dt} &= \frac{d}{dt}\left(\frac{a(1 \quad e^2)}{e\cos\theta + 1}\right) \\ &= \frac{a(1 \quad e^2)(e\sin\theta)}{(e\cos\theta + 1)^2}\frac{d\theta}{dt} \\ &= \frac{e\sin\theta}{e\cos\theta + 1}r\frac{d\theta}{dt} \end{aligned} \tag{16}$$

Substituting Equation (4) gives

$$\frac{dr}{dt} = \frac{h}{r}\frac{e\sin\theta}{e\cos\theta + 1}. \tag{17}$$

Applying both Equation (4) and Equation (17) to Equation (14) yields

$$
\begin{aligned}
E &= \frac{1}{2}m\left[\frac{h^2}{r^2}\frac{e^2\sin^2\theta}{(e\cos\theta+1)^2}+r^2\frac{h^2}{r^4}\right] \quad \frac{GMm}{r} \\
&= \frac{1}{2}m\left[\frac{h^2}{r^2}\left(\frac{e^2\sin^2\theta}{(e\cos\theta+1)^2}+1\right)\right] \quad \frac{GMm}{r} \\
&= \frac{mh^2}{2r^2}\frac{e^2+1+2e\cos\theta}{(e\cos\theta+1)^2} \quad \frac{GMm}{r}.
\end{aligned} \tag{18}
$$

Substituting Equation (11), it is calculated that

$$
\begin{aligned}
E &= \frac{GMm}{2r^2}\frac{a(1 \quad e^2)}{e\cos\theta+1}\frac{e^2+1+2e\cos\theta}{e\cos\theta+1} \quad \frac{GMm}{r} \\
&= \frac{GMm}{2r}\left(2+\frac{e^2 \quad 1}{e\cos\theta+1}\right) \quad \frac{GMm}{r} \\
&= \quad \frac{GMm}{2r}\frac{1 \quad e^2}{e\cos\theta+1} \\
&= \quad \frac{GMm}{2a}
\end{aligned} \tag{19}
$$

Thus, the energy of the planet is constant and so energy is conserved.

Law of Conservation of Energy. The energy of a planet orbiting around a star, denoted as $E = \frac{1}{2}mv^2 \quad \frac{GMm}{r}$, is conserved.

For instance, considering a hypothetical planet orbiting a star, the changes in kinetic and potential energy are calculated throughout its orbit, demonstrating the conservation principle.

4 CELESTIAL SPHERE

4.1 Latitude and Longitude on Earth

Understanding how longitude and latitude are measured from the perspective of the center of the Earth is crucial to the implementation of spherical trigonometry in the study of planetary trajectories.

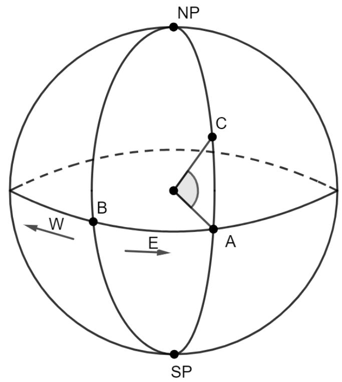

Figure 2: Latitude and Longitude Visualizations for an Arbitrary Point on Earth

In the Figure 2, the Earth's inner core sits at the center of the sphere with the North Pole pointing due north and the South Pole pointing due south. Point C represents an arbitrary point on Earth's surface, where $\overline{CA}$ denotes the arc length from that point to its corresponding point on the equator, or its latitude ϕ, and $\overline{BA}$ denotes the arc length from the point on the equator to a reference longitudinal line of 0, or its longitude l. From these definitions, the range of ϕ is $\frac{\pi}{2}$ ϕ $\frac{\pi}{2}$ while the range of l is π l π. Next, the model of a celestial sphere where the Sun orbits the Earth and the equivalent latitude and longitude in such a figure will be considered.

4.2 Right Ascension and Declination on a Celestial Sphere

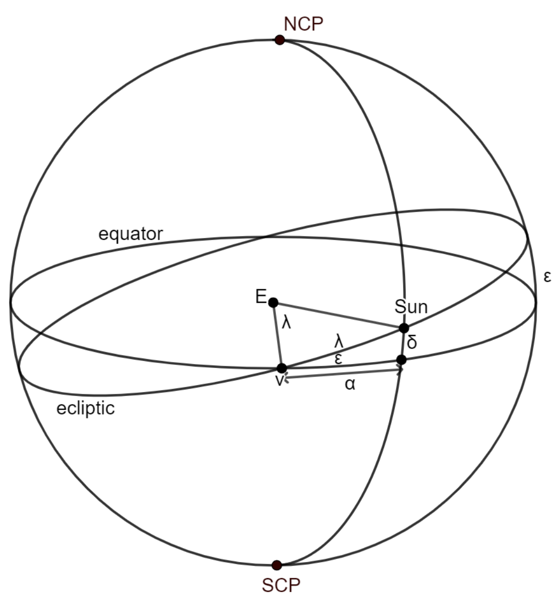

Figure 3: The Celestial Sphere around Earth Showing Key Celestial Coordinates and the Ecliptic

4.3 Defining the Celestial Sphere

While Kepler's Laws model the motion of celestial bodies through space, the motion of those bodies is perceived differently from Earth. In order to understand the relative positions of celestial bodies in the night sky, the concept of the Celestial Sphere is introduced.

The Celestial Sphere is an imaginary spherical shell of infinite radius centered at the Earth, upon which celestial bodies are projected. In Figure 3, the equator of the Earth aligns with the equator of the celestial sphere, or the celestial equator. Similarly, the North Celestial Pole faces due north and the South Celestial Pole faces due south.

The analog of longitude on the celestial sphere is formally known as the right ascension α and is measured west to east with the right ascension of the Vernal Equinox defined as 0°. The Vernal Equinox is the point at which the orbit of the sun, known as the ecliptic, intersects the celestial equator. The ecliptic forms an angle of $\epsilon = 23.5^\circ$ with the celestial equator. Right ascension is usually measured in hours and minutes, where 24 hours forms a full circle.

The analog of the latitude on the celestial sphere is the declination δ and is determined by the arc length from the sun's position to the celestial equator along a right ascension line.

4.4 Right Ascension and Declination of the Sun

One of the most important applications of spherical trigonometry to the position of bodies on the celestial sphere is determining the right ascension and declination of the sun. For any arbitrary position of the sun along its orbit, its ecliptic longitude λ is defined as the angle the sun forms with the Earth and the Vernal Equinox, with a range of $0^\circ \leq \lambda \leq 360^\circ$, as shown in Figure 3.

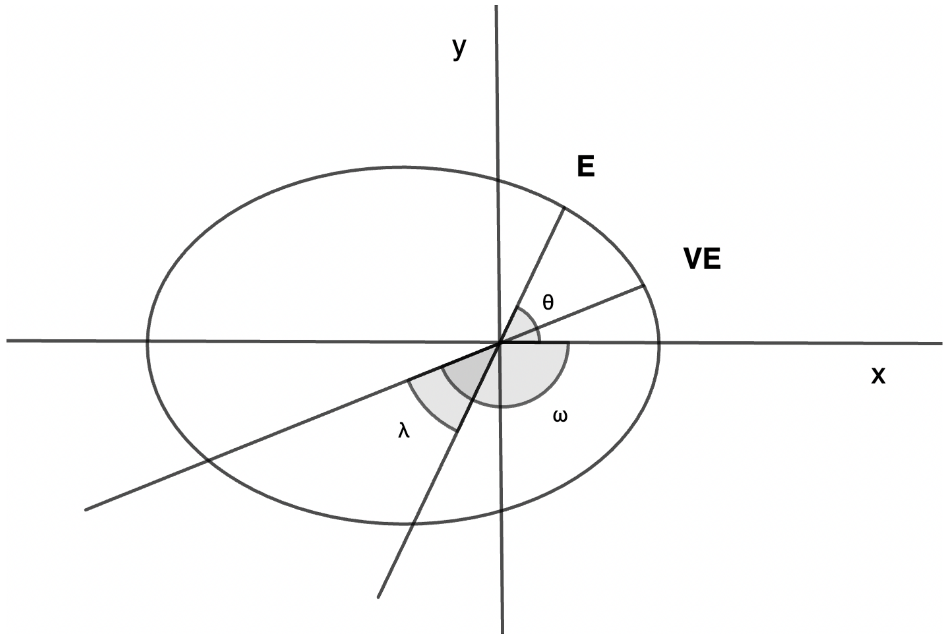

Figure 4: Derivation for Lambda

If θ is defined as the angle formed between the radius from the planet to its star and the radius from the planet's perhelion to the star, $\lambda = \theta + \omega - \pi$. The constant ω denotes the angle between the vernal equinox and the perihelion of the Earth. As shown in Appendix B.4, θ can be computed for any specific time of year t, and so λ can be computed as well.

Applying the spherical law of sines to the triangle formed by the sun, the Vernal Equinox, and the intersection of the sun's longitude and the celestial equator as in Figure 3 suggests that

$$\begin{aligned} \sin\delta &= \sin\lambda\sin\epsilon \\ \delta &= \sin^{-1}(\sin\lambda\sin\epsilon). \end{aligned} \tag{20}$$

Applying the law of cosines equation to the same spherical triangle yields the equation

$$\begin{aligned} \cos\lambda &= \cos\alpha\cos\delta \\ \alpha &= \cos^{-1}\left(\frac{\cos\lambda}{\cos\delta}\right). \end{aligned} \tag{21}$$

Equation (20) and Equation (21) can be used in tandem to calculate the right ascension and declination of the sun at any time of year. Since it is known that the Sun's right ascension ranges between 0h and 12h when its declination is positive (i.e. it crosses the celestial plane at the equinoxes), no expression for $\sin\alpha$ is needed to pin the exact quadrant in which α resides.

5 SATURN APPLICATION

5.1 Finding Lambda and Beta

To predict a planet's (in this case, Saturn's) location across the sky, the angular position of Saturn in its orbit with respect to perihelion (i.e. θ) must be translated to its corresponding variable in the celestial sphere. From observation, it is known that Saturn's path in the sky is only slightly tilted with respect to the ecliptic. Therefore, letting the ecliptic be the referential great circle for the planets in the celestial sphere rather than the celestial equator is convenient. So, to begin an analysis of its motion in the sky, Saturn's so-called ecliptic longitude (λ) and ecliptic latitude (β) must be derived from θ.

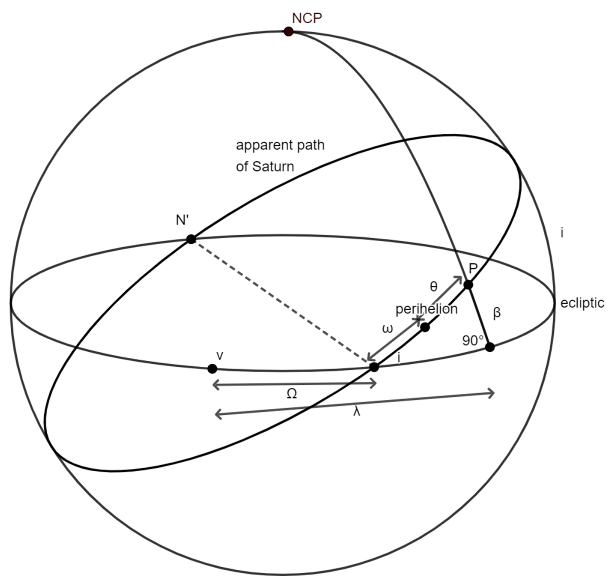

Figure 5: Saturn's Orbit on the Celestial Sphere

Using the laws of sines and cosines for spherical triangles, a relation between ecliptic longitude and latitude to Saturn's orbital parameters can be derived. Rewriting the law of sines and cosines for great spheres, the equations can be rewritten as:

$$\sin a \cos B = \sin c \cos b \quad \sin b \cos c \cos A$$

Thus, a triangle can be created:

Law of sines: $\frac{\sin\beta}{\sin i} = \sin(\omega + \theta)$

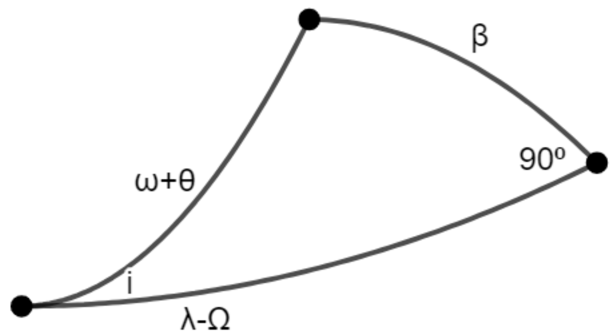

Figure 6: Spherical Triangle

Using the above conclusions, an analysis is now done of Saturn's orbit and the rising and setting times are predicted.

Saturn: $\Omega = 113.665$, $\omega = 339.392$, $i = 2.485$, $e = 0.0565$, $T = 10755.7$ days, and the perihelion is $11/29/32$. θ is calculated based on these values to be 254.429

$$\beta = \quad 2.0056$$

$$\sin(\omega+\theta)\frac{\cos i}{\cos\beta} = \sin(\lambda \quad \Omega)$$

$$\cos(\lambda \quad \Omega) = \frac{\cos(\omega+\theta)}{\cos\beta}$$

Note: An equation for cos and sin of $\lambda \quad \Omega$ is needed to find the explicit values for these constants

$$\lambda \quad \Omega = 233.795$$

$$\lambda = 347.46$$

$$\sin\delta = \cos\epsilon\sin\beta + \sin\epsilon\cos\beta\sin\lambda$$

$$\sin\beta = \cos\epsilon\sin\delta \quad \sin\epsilon\cos\delta\sin\alpha$$

$$\cos\alpha = \frac{\cos\beta\cos\lambda}{\cos\delta}$$

$$\delta = \quad 6\ 48^{0}44.64\text{”}$$

$$\alpha = 23\ 17^{0}2.4\text{”}$$

5.2 Calculating the Rise and Set of Saturn

After calculating the right ascension, α, and the declination, δ, of Saturn, it is possible to calculate the time and duration during which it is visible in the sky at a specific latitude on Earth. First, consider a vantage point on Earth with a latitude $\phi = 40.76$. Applying the spherical Law of Cosines, it is obtained that $\cos H = \quad \tan\phi\tan\delta$. Where H is the hour angle, which measures the time since a celestial object was last on the local meridian. It is used to calculate the rise and set times of celestial bodies.

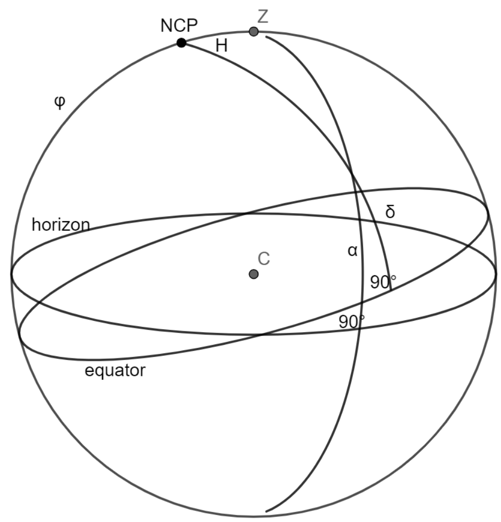

Figure 7: Diagram of H

As previously proven for 7/22/24, $\delta = \quad 6.8124$, which gives $H = 95.899$. Since H is defined as the duration from when Saturn first rises above the horizon to its highest point in the sky, doubling H will give the total duration during which Saturn is visible. Converting to hours and minutes, it is determined that Saturn will be visible for $2H = 11h13m$.

Next, it is possible to find the time at which Saturn is at its highest point by comparing its right ascension to the right ascension of the sun. The difference of these right ascensions will produce the phase shift between the two orbits, and thus converting to hours and minutes will determine the time at which Saturn reaches its highest point. The difference is $\alpha_s \quad \alpha = 23h17m \quad 8h10m = 15h07m$, where α_s is the right ascension of Saturn. Knowing that the sun is at its highest point at 1:00 PM EST (noon of daylight time), adding this acquired phase shift, 15 hours and 7 minutes, provides that Saturn will reach its highest point at 4:07 AM EST on 7/23/24.

Finally, to calculate the time of the rise and set of Saturn, H is added and subtracted. H: $4h07m \quad H = [22h30m, 9h44m]$. It is thus determined that Saturn will rise at 10:30 PM EST on 7/22/24 and will set at 9:44 AM EST on 7/23/24. This value was found to be very close to the true value.

5.3 *Predicting Saturn's Orbit with Machine Learning Models*

The formulas for right ascension and declination derived from Kepler's laws are not perfect. Failing to account for the gravitational influences from other celestial bodies and the relativistic effects and non-uniform mass distributions within celestial objects, the predicted value determined by the formulas slightly deviate from actual telescopic measurements.

5.4 *Error Calculations*

To begin, thousands of data points using constants for each planet [5] were generated beginning from January 1st, 2014 to present day to predict the location of Saturn on the celestial sphere using Kepler's laws as discussed previously. These values were then compared with the values from an ephemeris [3] and NASA's telescopic data to identify the percent error between the sets of data.

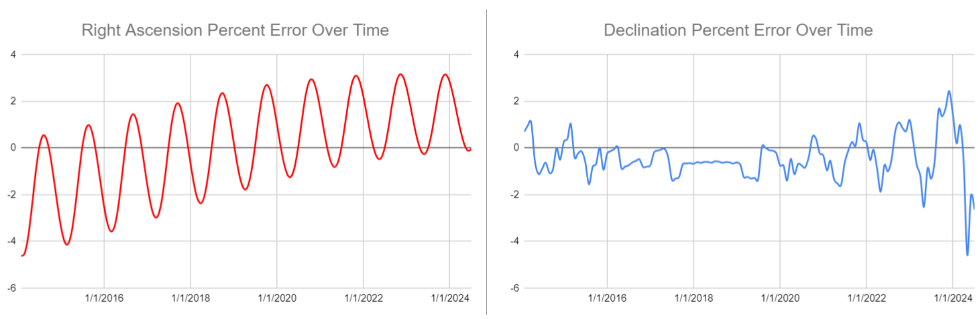

Figure 8: Percent Error of the Right Ascension and Declination Calculated from Comparisons between Ephemeris Data and Kepler's Laws

It is seen in Figure 7 that the error in the calculated right ascension and declination values with Kepler's laws vary very slightly from the ephemeris values, by single digit numbers less than 5. While Kepler's laws provide a great approximation of planetary motion, the data points generated from these laws can vary slightly from the actual observed values of planets in the sky due to several additional factors, which explains the variations seen in the figure above. One factor is the gravitational perturbations caused by other celestial bodies such as large planets, which Kepler's laws do not account for. Additionally, the effects of general relativity, which Kepler's classical laws do not incorporate, can cause slight deviations in planets' orbits, especially noticeable in the perihelion precession of planets.

Both of these graphs exhibit very clear sinusoidal patterns. Right Ascension has a steady and slow decrease of amplitude and a rise in peaks and troughs of the sinusoidal waves. Meanwhile, the Declination has a decrease and then an increase in the amplitude of the waves. These periodic patterns show the shortcomings of the Kepler's laws in predicting the actual declination and right ascension of celestial objects. Likely caused by other planets and gravitational forces, Saturn oscillates from these very slight forces causing this trend for right ascension. This seems to be the same case for Declination where approximately halfway through Saturn's orbit (2018), Saturn barely oscillates with respect to its expected path. Then, Saturn starts to oscillate increasingly over the next part of its orbit.

5.5 Implementation of Machine Learning

After quantifying the error between the calculated predictions of Saturn's right ascension and declination to its actual values, machine learning algorithms were implemented to find the relationship in the error between the predicted and actual values. First, data points were generated for each day between 01/01/2014 to 05/31/2024 using the formulas derived from Kepler's laws. Next, the actual celestial coordinates were obtained of Saturn for each day from the same ephemeris. Two linear regression models were trained that determined linear relationships between the values generated from Kepler's formulas and actual ephemeris data, one for right ascension and the other for declination. In essence, these models attempt to linearize and predict the error between the calculated coordinates and actual coordinates of Saturn. When testing the regression models on a set of test data, a R^2 of 0.927 was obtained for right ascension and a R^2 of 0.889 was obtained for declination. Plotting the predicted values from the linear regression models against the actual ephemeris values in the test data set yielded the graphs below. The residuals between the models' predicted values and the actual ephemeris data are also plotted.

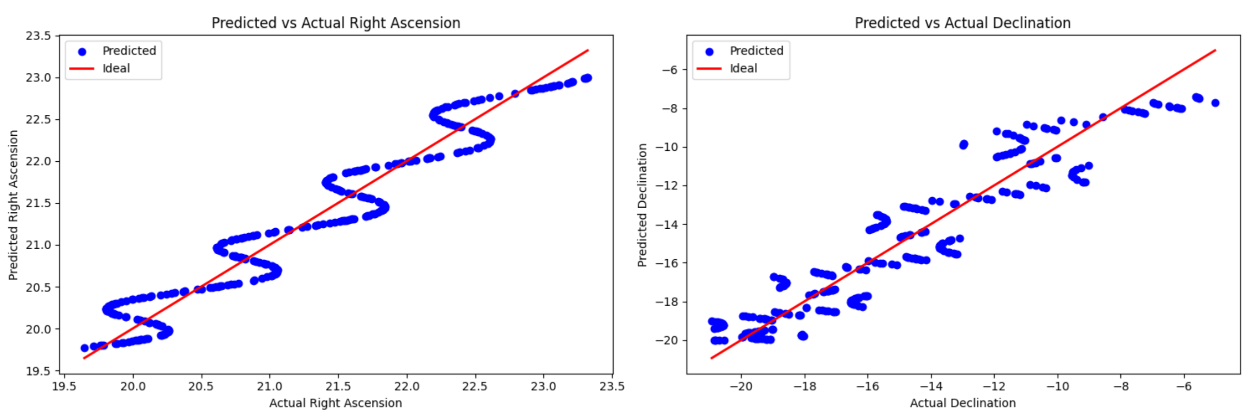

Figure 9: Regression Models Predictions vs Actual Ephemeris Data

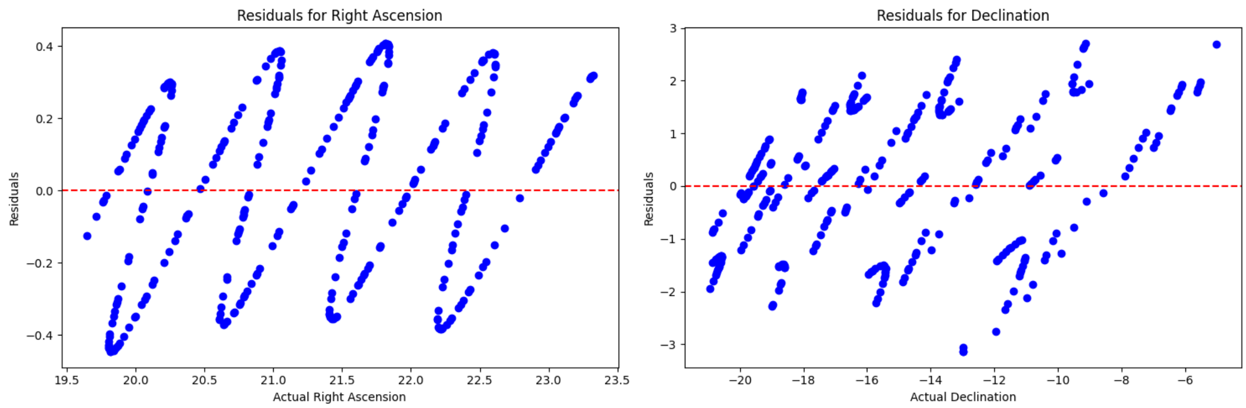

Figure 10: Regression Models Residual Plots

The red lines in the scatter plots represent the ideal scenario where the regression models are completely accurate in predicting the right ascension and declination values of Saturn based on the calculated coordinates in the test data set. The wavy, repetitive pattern in the scatter and residual plots indicate that the linear regression models are not the most accurate. This is because they are not complex enough to capture the underlying periodic, or cyclic, trends in Saturn's behavior.

To account for these complexities, more robust and adaptive random forest models were implemented. Using the same dataset of generated values from the formulas and Saturn's actual right ascension and declination coordinates from the ephemeris, two random forest models were trained and tested to find the relationships in the error between the calculated data points and the actual data points. The right ascension model yielded a R^2: 0.999 and the declination model resulted in a R^2: 0.997. Plotting the models' predicted values compared to the actual ephemeris values in the test data set yielded the graphs below. The residuals between the models' predicted values and the actual ephemeris data are once again plotted as well.

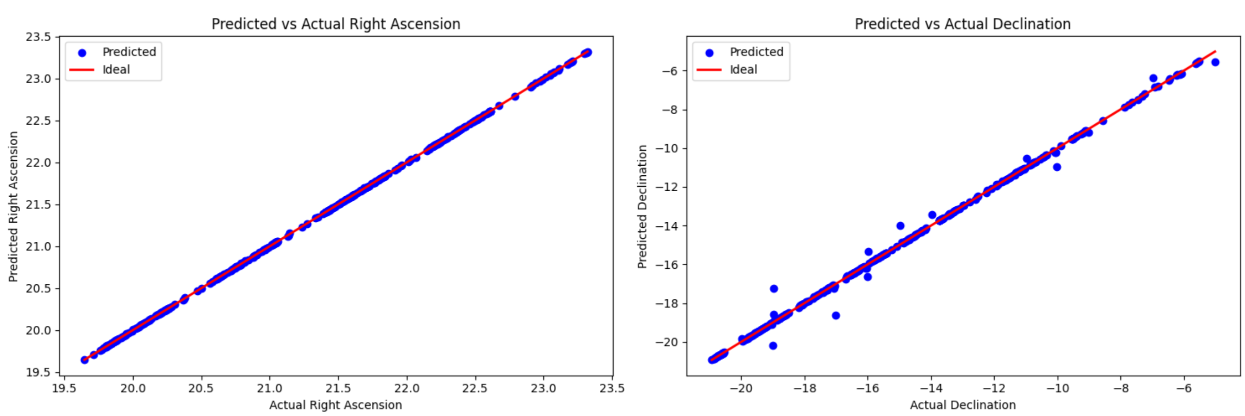

Figure 11: Random Forest Models Predictions vs Actual Ephemeris Data

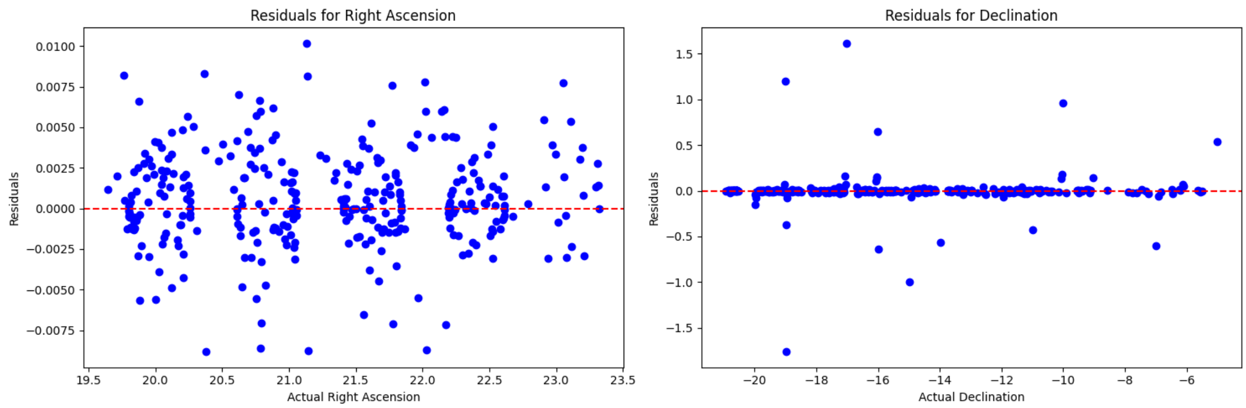

Figure 12: Random Forest Models Residual Plots

The random forest models were extremely successful at predicting error and thus the actual celestial coordinates of Saturn based on the calculated values from the derived formulas. The scattered pattern in the residual plots also verify the applicability of the models.

5.5.1 One-Class Support Vector Machines Anomaly Detection

To better identify anomalies within Saturn's orbit over a 21-year period from 2003 to now, telescopic data from the European Space Agency (ESA) [4] was used in conjunction with the One-Class Support Vector Machines model. The data was pre-processed to convert both the right ascension and declination from the telescopic data to degrees, and any values which were found to be displayed incorrectly were removed from the data set. The data was then normalized, adjusting the right ascension and declination values to a common scale which would then allow the machine learning model to not be disproportionately affected by the different scales. The normalization of the data accounts for how right ascension and declination have different ranges which could dominate the machine learning model.

By using One-Class SVM, patterns in data were able to be identified using the normalized input, to then identify the outliers that are classify as anomalies. The Radial Basis Function Kernel was used to capture these patterns in the data. The hyper parameter ν was set to 0.001, meaning that it was anticipated that about 0.1% of the normalized data points were anomalies. The model learns the boundary that encompasses most of the normal data points within it, while anything outside would be classified as an anomaly. A value +1 was assigned if it was classified as normal, and a value of -1 was assigned if it was classified as an anomaly. Figure 13 helps visualize these anomalies, which could potentially indicate unusual celestial events or observational or recording error, like incorrect time stamps or equipment malfunctions. The code for the model can be found in Appendix C.

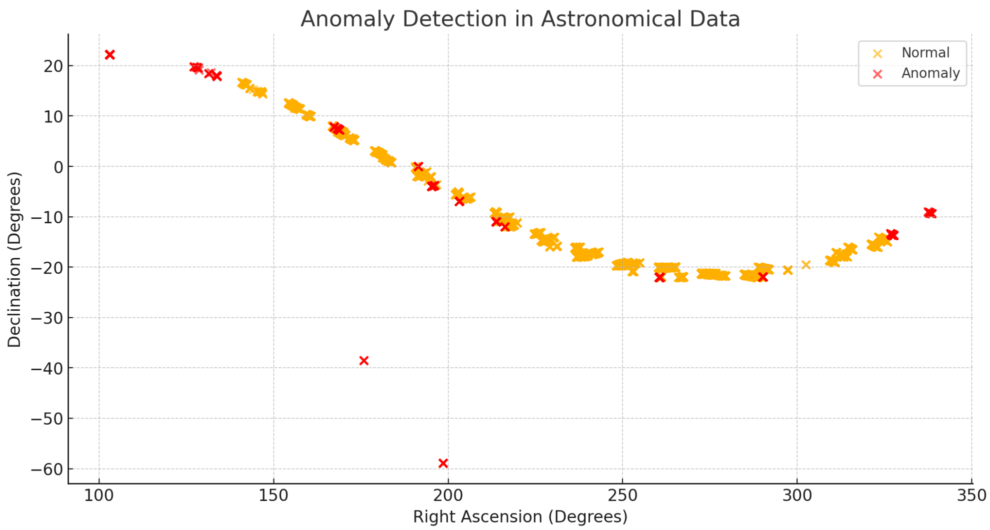

Figure 13: Results of the One-Class SVM

6 CONCLUSION

In this paper, Newton's universal law of gravitation and Kepler's laws of planetary motion were used to calculate and predict the positions of Earth and Saturn, but this methodology can be expanded onto other celestial objects. Mathematical derivations and represent geometries were performed that prove the applicability of classical mechanics to modern astronomy. The spherical trigonometry and ellipse geometry explorations provide further validation of the accuracy of these laws.

Additionally, by incorporating machine learning models, including linear regression and random forest, the precision of the predictions related to the orbit of Saturn increased. More importantly, the random forest model was very successful in terms of reducing error regarding the predictions, proving that classical astronomical theories can go alongside state-of-the-art techniques for data analysis.

Furthermore, the use of One-Class Support Vector Machines for anomaly detection proved effective in identifying anomalies within Saturn's orbit over a 21-year period. This technique can help in pinpointing unusual celestial events or potential observational errors, adding another layer of accuracy and reliability to the study.

The study thus reiterates the resilience of Kepler's laws while conceding that classical mechanics may become very basic in regard to gravitational perturbations and relativistic effects. This work could be furthered by adding other, more complex models in such a manner that would achieve improved accuracy in prediction.

These results bear significance not only on practical issues involving the understanding of planetary motion but also emphasize how interdisciplinary approaches can stretch the envelope of the present status of astronomy.

7 REFERENCES

References

[1] Newton, I. *Philosophiæ Naturalis Principia Mathematica*. London: Royal Society, 1687.

[2] Smart, W. M. *Textbook on Spherical Astronomy*. Cambridge: Cambridge University Press, 1977.

[3] In-The-Sky.org. Ephemeris. *In-The-Sky.org*. https://in-the-sky.org/ephemeris.php. Accessed 2024 Jul 17-26.

[4] European Southern Observatory. 2023. *ESO Science Archive Facility*. https://archive.eso.org/eso/eso_archive_main.html. Accessed 2024 Jul 17-26.

[5] *Sol Planetary System Data*. https://www.princeton.edu/ willman/planetary_systems/Sol/. Accessed 2024 Jul 17-26.

A SPHERICAL TRIGONOMETRY

A.1 Spherical Triangles

Beginning with a sphere of radius R and three points, a spherical triangle is formed. The sphere is oriented such that there is a plane tangent to one point. Points are then projected from the center of the sphere through the two other points onto the plane, forming a Euclidean triangle on the plane.

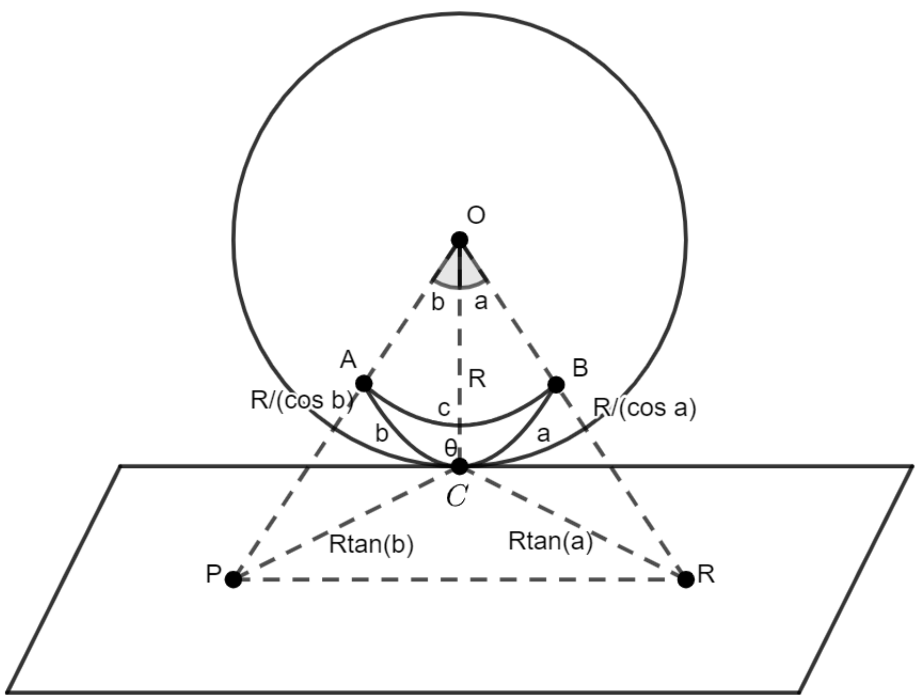

Figure 14: A Spherical Triangle, Labeled

Where A, B, and C are expressed in degrees, describing the angle formed by the two tangents at each respective vertex of the spherical triangle. a,b, and c are also expressed in degrees, describing the angle at the origin between each pair of vertices.

A.2 Spherical Law of Cosines

OP can be expressed as $\frac{R}{\cos(b)}$ and OR as $\frac{R}{\cos(a)}$. CP can be expressed as $R\tan(b)$ and CR as $R\tan(a)$. Using the Pythagorean Theorem, it can said that

$$PR^2 = (\frac{R}{cos(a)})^2 + (\frac{R}{\cos(b)})^2 \quad \frac{2r^2\cos(c)}{\cos(a)\cos(b)}$$

Applying the Law of Cosines on CPR, an alternate expression can be obtained for PR^2.

$$PR^2 = (R\tan(b))^2 + (R\tan(a))^2 \quad 2(R\tan(b))(R\tan(a))(\cos(C))$$

After setting these two expressions for PR^2 equal to one another and factoring out R^2, $cos(c)$ can be isolated.

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C)$$

Which is known as the Spherical Law of Cosines. It can observed that this law is similar in essence to the Euclidean Law of Cosines.

A.3 Spherical Law of Sines

Next, the Law of Sines in Spherical Geometry can be derived. Starting with the Spherical Law of Cosines, it can be said that

$$\cos(C) = \frac{\cos(c) \quad \cos(a)\cos(b)}{\sin(a)\sin(b)}$$

By the Pythagorean Identity,

$$\sin^2(C) = 1 \quad (\frac{\cos(c) \quad \cos(a)\cos(b)}{\sin(a)\sin(b)})^2$$

So,

$$\frac{\sin^2(C)}{\sin^2(c)} = \frac{1 \quad (\frac{\cos(c) \quad \cos(a)\cos(b)}{\sin(a)\sin(b)})^2}{\sin^2(c)}$$

Simplifying,

$$\frac{\sin^2(C)}{\sin^2(c)} = \frac{1 \quad \cos^2(a) \quad \cos^2(b) \quad \cos^2(c) + 2\cos(a)\cos(b)\cos(c)}{\sin^2(a)\sin^2(b)\sin^2(c)}$$

It can be observed that this is a symmetric function for a,b, and c.

$$\frac{\sin^2(A)}{\sin^2(a)} = \frac{\sin^2(B)}{\sin^2(b)} = \frac{\sin^2(C)}{\sin^2(c)}$$

[21-23]

Thus, the Spherical Law of Sines is derived, which, too, is reminiscent of its Euclidean counterpart.

B ELLIPSE GEOMETRY

B.1 Preliminary

An ellipse is uniquely determined by two foci F_1 and F_2 and a constant a. Specifically, the ellipse is defined as the locus of points P such as the sum of the distance PF_1 and the distance PF_2 is equal to $2a$. If line segment $\overline{F_1F_2}$ lies on the x-axis of a Cartesian coordinate system and the origin f is the midpoint of segment $\overline{F_1F_2}$, it can be shown that the ellipse is represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the distance F_1F_2 is defined as $2c = 2\sqrt{a^2 - b^2}$.

The length a is known as the semi-major axis of the ellipse and b is known as the semi-minor axis. The eccentricity of the ellipse is defined as $e = c/a$. This representation of an ellipse in Cartesian coordinates can be converted into a polar form, which would make it easier to work with distances from the center and central angle.

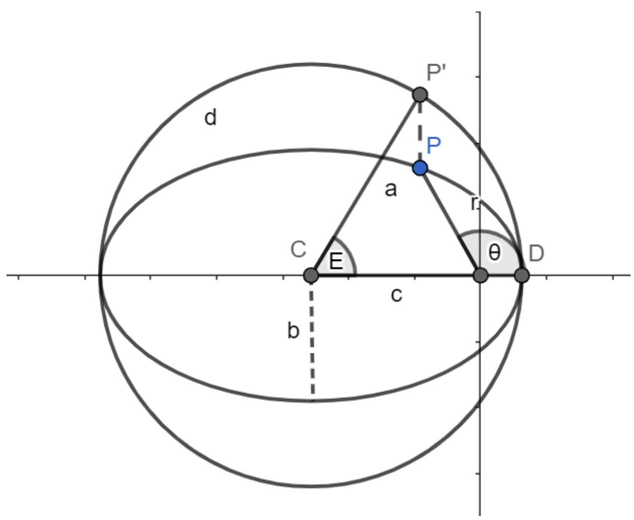

Figure 15: Ellipse and Circumscribed Circle Diagram

B.1.1 An Ellipse in Polar Coordinates

Translating the ellipse such that one focus of the ellipse lies at the origin, it is calculated that

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (22)$$

[21-24]

Converting to polar coordinates by substituting $x = r\cos\theta$ and $y = r\sin\theta$,

$$\begin{aligned}
&\frac{(r\cos\theta + c)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1 \\
&\frac{(r\cos\theta + ea)^2}{a^2} + \frac{(r\sin\theta)^2}{a^2(1 \quad e^2)} = 1 \\
&(1 \quad e^2)(r\cos\theta + ea)^2 + (r\sin\theta)^2 = a^2(1 \quad e^2) \\
&(1e^2\cos^2\theta)r^2 + 2ae\cos\theta(1 \quad e^2)r \quad a^2(1 \quad e^2)^2 = 0
\end{aligned} \tag{23}$$

Thus, r is equal to

$$r = \tag{24}$$

$$\frac{2ae^3\cos\theta \quad 2ae\cos\theta \quad \sqrt{(2ae^3\cos\theta \quad 2ae\cos\theta)^2 \quad 4(e^2\cos^2\theta \quad \sin^2\theta \quad \cos^2\theta)(a^2 \quad 2a^2e^2 + a^2e^4)}}{2(e^2\cos^2\theta \quad \sin^2\theta \quad \cos^2\theta)}$$

$$\frac{2ae^3\cos\theta \quad 2ae\cos\theta \quad \sqrt{4a^2e^2\cos^2\theta(e^2 \quad 1)^2 \quad 4a^2(e^2\cos^2\theta \quad 1)(1 \quad 2e^2 + e^4)}}{2(e^2\cos^2\theta \quad 1)}$$

$$\frac{ae\cos\theta(e^2 \quad 1) \quad (e^2 \quad 1)a\sqrt{e^2\cos^2\theta \quad (e^2\cos^2\theta \quad 1)}}{(e^2\cos^2\theta \quad 1)}$$

$$\frac{ae\cos\theta(e^2 \quad 1) \quad (e^2 \quad 1)a}{(e\cos\theta \quad 1)(e\cos\theta + 1)}$$

$$\frac{a(e^2 \quad 1)(e\cos\theta \quad 1)}{(e\cos\theta \quad 1)(e\cos\theta + 1)}$$

Since r has to always be positive, all the parts of the fraction can be analyzed.

$$r = \frac{a(e^2 \quad 1)(e\cos\theta \quad 1)}{(e\cos\theta \quad 1)(e\cos\theta + 1)}$$

$$r = \frac{a(1 \quad e^2)}{1 + e\cos\theta} \tag{25}$$

B.2 Connecting θ and E

B.2.1 Finding $\cos E$

The same x value of P is projected onto the outer circle on the point P'.
$a\cos E$ is the x value of P' with respect to the center of the circle and ellipse. Thus:

$$a\cos E = c + x$$

$$a\cos E = ae + r\cos\theta \tag{26}$$

[21-25]

Substituting the relationship for r.

$$cosE = \frac{\cos\theta + e}{1 + e\cos\theta}$$

B.2.2 Finding $\sin E$

Nonetheless, the same can be done with $\sin$. The only problem is that the value of the y of P' is not solved for. This can be found by finding the relationship of the y of P' and P at any given time. This can be converted to rectangular form:
Circle: $x^2 + y'^2 = a^2$
Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$y' = \sqrt{a^2 \quad x^2}$$

$$y = \sqrt{b^2 \quad \frac{x^2b^2}{a^2}}$$

$$\frac{\sqrt{a^2 \quad x^2}}{\sqrt{b^2 \quad \frac{x^2b^2}{a^2}}} = \frac{1}{b\sqrt{\frac{1}{a^2}}} = \frac{a}{b}$$

Thus:

$$a\sin E = r\sin\theta\frac{a}{b}$$

B.2.3 Geometric Relationship between circle and ellipse

Geometrically, rsinθ is equal to the y-value of P. Meanwhile, asinE is equal to the y-value of P'. The ratio between these two is found by expressing the circle and ellipse in a rectangular fashion:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 + y'^2 = a^2$$

Note: The ellipse and circle is shifted to be centered at the origin because the ratio will remain the same.

$$y^2 = \frac{a^2b^2 \quad b^2x^2}{a^2}$$

$$y'^2 = a^2 \quad x^2$$

$$\sqrt{\frac{y^2}{y'^2}} = \sqrt{\frac{a^2b^2 \quad b^2x^2}{a^2(a^2 \quad x^2)}}$$

$$\frac{y}{y'} = \frac{b}{a}$$

[21-26]

Now, going back to the $r \sin\theta$ and $a \sin E$

$$r \sin\theta = \frac{b}{a} a \sin E$$

$$r \sin\theta = b \sin E \tag{27}$$

B.2.4 Finding $tan \frac{E}{2}$

First, it can be considered that n is some function or factor that maps $\tan\left(\frac{E}{2}\right)$ to $\tan\left(\frac{\theta}{2}\right)$.

$$\tan\left(\frac{E}{2}\right) = n \tan\left(\frac{\theta}{2}\right)$$

$$\tan^2\left(\frac{E}{2}\right) = \frac{\sin^2\left(\frac{E}{2}\right)}{\cos^2\left(\frac{E}{2}\right)}$$

Using Half-Angle identities it is derived that

$$\tan^2\left(\frac{E}{2}\right) = \frac{1 \quad \cos E}{1 + \cos E}$$

$$\tan^2\left(\frac{\theta}{2}\right) = \frac{1 \quad \cos\theta}{1 + \cos\theta}$$

Substituting the relationship for $\cos E$ and dividing both relationships yields

$$n^2 = \left(\frac{1 \quad \left(\frac{\cos(\theta)+e}{1+e\cos(\theta)}\right)}{1 + \left(\frac{\cos(\theta)+e}{1+e\cos(\theta)}\right)} \right) \frac{1+\cos(\theta)}{1 \quad \cos(\theta)}$$

$$n = \sqrt{\frac{1 \quad e}{1+e}}$$

B.3 *Consider* $M = E \quad e \sin E$*, then* $\frac{dM}{dt} = cst$

B.3.1 Reminder of Formulas

$$\cos E = \frac{\cos\theta + e}{1 + e\cos\theta}$$
$$\frac{d\theta}{dt} = \frac{h}{r^2}$$
$$r\sin\theta = b\sin E$$
$$r = \frac{a(1 \quad e^2)}{1 + e\cos\theta}$$

B.3.2 Proof

$$1 + e\cos\theta = \frac{a(1 \quad e^2)}{r}$$
$$\frac{d\theta}{dt} e\sin\theta = \quad r^{\ 2} a(1 \quad e^2)\frac{dr}{dt}$$
$$\frac{h}{r^2} e\sin\theta = \quad r^{\ 2} a(1 \quad e^2)\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{eh\sin\theta}{a(1 \quad e^2)}$$

Now, the derivative of $r\sin\theta = b\sin E$ is taken.

$$\frac{dr}{dt}\sin\theta + \frac{d\theta}{dt} r\cos\theta = \frac{dE}{dt} bcosE$$

$$\frac{eh\sin^2\theta}{a(1 \quad e^2)} + \frac{h}{r^2} rcos\theta = \frac{dE}{dt} a^{p}\overline{1 + e^2}\frac{\cos\theta + e}{1 + e\cos\theta}$$
$$\frac{eh\sin^2\theta}{a(1 \quad e^2)} + \frac{h(1 + e\cos\theta)}{a(1 \quad e^2)}\cos\theta = \frac{dE}{dt} a^{p}\overline{1 + e^2}\frac{\cos\theta + e}{1 + e\cos\theta}$$
$$\frac{h(1 + e\cos\theta)}{a^2(1 \quad e^2)^{\frac{3}{2}}} = \frac{dE}{dt}$$

Now the derivative of $M = E \quad e\sin E$ is taken and $\frac{dE}{dt}$ that was found is substituted.

$$\frac{dM}{dt} = \frac{dE}{dt}(1 \quad e\cos E)$$

$$\frac{dM}{dt} = \frac{h(1+e\cos\theta)}{a^2(1-e^2)^{\frac{3}{2}}}(1 - e\frac{\cos\theta + e}{1+e\cos\theta})$$

$$\frac{dM}{dt} = \frac{h(1+ecos\theta)}{a^2(1-e^2)^{\frac{3}{2}}}(\frac{1+e\cos\theta - e\cos\theta - e^2}{1+e\cos\theta})$$

$$\frac{dM}{dt} = \frac{h}{a^2(1-e^2)^{\frac{1}{2}}} = \frac{h}{ab}$$

This is rewritten so that $\frac{dM}{dt}$ is in terms of time. It is known that:

$$T^2 = \frac{4\pi^2 a^2 b^2}{GMa(1-e^2)} = \frac{4\pi^2 a^2 b^2}{h^2}$$

$$T = \frac{2\pi ab}{h}$$

$$\frac{dM}{dt} = \frac{h}{ab} = \frac{2\pi}{T} \tag{28}$$

B.4 Connecting t with θ

A relationship is established from $t \rightarrow M \rightarrow E \rightarrow \theta$ where $\frac{dM}{dt}$ is a constant and can be used to further calculate the position of celestial objects based on an initial position and measuring several constants.

$$\frac{dM}{dt} = \frac{2\pi}{T}$$

$$M = E - e\sin E$$

$$\tan\frac{E}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}$$

Using these equations, the angle between the Sun and celestial objects is easily understood. Consider the angle between Earth and the Sun on July 27th, 2126. The eccentricity of Earth's orbit $(e = 0.017)$ and the period of revolution for Earth to be $(T = 365.25)$.

$$\int dM = \int_0^{37451} \frac{2\pi}{T}\, dt$$

$$M = 644.248$$

Solving the relationship between M and E via analytical methods is not a clear or straightforward procedure:

$$644.248 = E - 0.017\sin E$$

However, E can be approximated by observing that the $0.017\sin E$ term is near zero, thereby making a first approximation of $E = 644.248$. This value can then be re-substituted as the argument to the above sine function in the original equation to find a new value for E. This process can be iterated to obtain any desired degree of accuracy:

$$E = 644.244$$

Finally substituting the values for E and e

$$\tan\left(\frac{644.244}{2}\right) = \sqrt{\frac{1+0.017}{1\quad 0.017}}\tan\left(\frac{\theta}{2}\right)$$

$$\theta = 192.674$$

C ONE CLASS SVM CODE

The data used can be found here:

https://www.kaggle.com/datasets/adityakiru/saturn-right-ascention-and-declination/

```
import pandas as pd
import numpy as np
import re
from sklearn.preprocessing import StandardScaler
from sklearn.svm import OneClassSVM
import matplotlib.pyplot as plt

file_path = 'saturndata.csv'
data = pd.read_csv(file_path)

def convert_right_ascension(ra_str):
    match = re.match(r_a, ra_str)
    if match:
        hours = int(match.group(1))
        minutes = int(match.group(2))
        seconds = float(match.group(3))
        decimal_degrees = 15 * (hours + minutes / 60 + seconds / 3600)
        return decimal_degrees
    return None

def convert_declination(dec_str):
    match = re.match(r_a, dec_str)
    if match:
        degrees = int(match.group(1))
        arcminutes = int(match.group(2))
        arcseconds = float(match.group(3))
        decimal_degrees = degrees + arcminutes / 60 + arcseconds / 3600
        return decimal_degrees
    return None
```

```python
data['Right Ascension (Degrees)'] = data['Right Ascention'].apply(
    convert_right_ascension)
data['Declination (Degrees)'] = data['Declination'].apply(convert_declination)
data.dropna(subset=['Right Ascension (Degrees)', 'Declination (Degrees)'],
    inplace=True)

features = data[['Right Ascension (Degrees)', 'Declination (Degrees)']]
scaler = StandardScaler()
features_normalized = scaler.fit_transform(features)
ocsvm = OneClassSVM(kernel='rbf', gamma='auto', nu=0.001)
ocsvm.fit(features_normalized)
predictions = ocsvm.predict(features_normalized)
data['Anomaly'] = predictions
normal_data = data[data['Anomaly'] == 1]
anomalous_data = data[data['Anomaly'] == -1]
plt.figure(figsize=(12, 6))
plt.scatter(normal_data['Right Ascension (Degrees)'], normal_data['Declination
     (Degrees)'], label='Normal', alpha=0.6)
plt.scatter(anomalous_data['Right Ascension (Degrees)'], anomalous_data['
    Declination (Degrees)'], color='red', label='Anomaly', alpha=0.6)
plt.title('Anomaly Detection in Astronomical Data')
plt.xlabel('Right Ascension (Degrees)')
plt.ylabel('Declination (Degrees)')
plt.legend()
plt.grid(True)
plt.show()
num_anomalies = (data['Anomaly'] == -1).sum()
print(f'Number of anomalies detected: {num_anomalies}')
```